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Effective constitutive properties of a disordered elastic solid medium via the strong-fluctuation approach

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A general scheme is developed for estimating the effective constitutive properties of a randomly disordered elastic solid medium. The presented methodology, already known in electromagnetics and acoustics, is based on renormalizing the conventional equations of motion. The resulting equations lend themselves to an approximate averaging procedure, which holds for strong fluctuations in the constitutive properties of the disordered medium provided the renormalization constants are chosen appropriately and the length-scales of the random perturbations are small. As an example, the homogenization of anisotropic spherical inclusions randomly dispersed in an isotropic host medium is considered, and the effective Lamé constants of the homogenized disordered medium evaluated and discussed.

Keywords: composite materials; disordered media; homogenization theory; strong-fluctuation theory; renormalization

1. Introduction

This paper is concerned with the homogenization of a linear elastic solid medium with stochastic variation of constitutive properties (CPs) at short length-scales. The propagation of suitably averaged time-harmonic elastodynamic fields in such a disordered medium occurs in a manner characteristic of a homogeneous medium. The CPs of the latter medium represent the effective constitutive properties (ECPs) of the homogenized disordered medium (HDM). Because of obvious applications, particularly for polycrystalline solids and composite materials, the problem of predicting the ECPs of HDMs has long been of interest to researchers. As a result, a large body of literature on this topic exists, from which we shall cite here only the most relevant works.

Most commonly, a disordered medium is modelled either as a homogeneous matrix medium containing a distribution of discrete inclusions with random positions and orientations, or as a random continuum with statistically fluctuating CPs. Evidently, the first model may be regarded as a special case of the second, though the two models can lead to different homogenization approaches.

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In the discrete scatterer model, the ECP estimation requires consideration of scattering by individual inclusions. This is usually supplemented by suitable assumptions on the interaction among inclusions, such as Foldy's approximation (Foldy 1945) or the more refined quasi-crystalline approximation (QCA) of Lax (1952), to make the many-body problem tractable. The QCA requires knowledge not only of the inclusion volume fraction but also of higher-order statistical information (Willis 1980*b*; Tsang & Kong 1981*a*; Varadan *et al*. 1989), specifically, the two-particle distribution function, which is not often possible in practice. Self-consistent approaches look simpler in this regard, since they do not make use of many-particle distribution functions (Budiansky 1965; Hill 1965; Middya *et al*. 1985; Sabina & Willis 1988; Sabina *et al*. 1993; Smyshlyaev *et al*. 1993). The relative lack of statistical information, however, may have deleterious effects on the adequacy of these approaches.

In the random continuum model, the CPs of a disordered medium are supposed to be random functions with known statistics, and the ECPs of the HDM are estimated from a knowledge of the mean field. This can be done in several ways. One practical though crude method is to estimate the ECPs with the assumption of either homogeneous strain (Voigt 1928) or homogeneous stress (Reuss 1929). A rigorous variational approach for bounding the ECPs of disordered media on the basis of limited statistical information was advanced by Hashin & Shtrikman (1962), Kröner (1977), Gairola & Kröner (1981), Willis (1981), Talbot & Willis (1982*a*–*c*) and Hashin (1983), among others; see also Torquato's review of bounds on elastostatic ECPs (Torquato 1991). The variational approach has, of course, little use when the bounds are widely separated.

Also for the random continuum model, a multiple-scattering theory invoking the solution of a Dyson-type integral equation for the mean field is often used, the kernel of the integral equation being expressed through the correlation functions of random CP perturbations. In the framework of this theory, (i) the now widely used bilocal approximation for estimating the ECPs of a polycrystalline medium with orientationally disordered grains was introduced by Lifshitz & Rosenzveig (1946, 1951) and by Lifshitz & Parkhomovski (1948, 1950) in elastostatics and ultrasonics, respectively; and (ii) a full perturbation series solution for the ECPs was developed by Kudinov *et al*. (1975) and Shermegor (1977, p. 157) with the fluctuating parts of the CPs identified through a small parameter. A satisfying feature of this approach is that it enables one, at least in principle, to ascertain the ECPs of an HDM to any desired order of accuracy in the small parameter used. Also, even the lowest-order (i.e. the bilocal) approximation of this method is capable of accounting for the microstructure of a disordered medium and reveals attenuation of the mean field due to multiple-scattering processes (which is not evinced by many other theories, but see Shanker & Lakhtakia (1993) and Prinkey *et al*. (1994)). The results of Lifshitz & Parkhomovski (1948, 1950) were extended by Atthey (1985) to the case of disordered materials with small textural anisotropy. A refinement of the bilocal approximation through the causality principle was attempted by Beltzer & Brauner (1985). Clearly, this approach is effective only if the CP fluctuations are small and perturbation theory can be used.

Of other modifications of multiple-scattering theory for elastodynamic homogenization, we single out a renormalized formulation reported by Chigarev (1980). The Lippman–Schwinger equation for the strain tensor of a disordered medium was reformulated in terms of a new field variable by extracting the pointwise singularity of an appropriate fourth-rank Green tensor. The resultant singular integral equation with a spherical exclusion region was averaged, assuming strongly isotropic fluctuations, to yield non-local ECPs at arbitrary wavelengths of external time-harmonic stimuli. Renormalization indeed has spawned strong-fluctuation theories, for electromagnetics (Finkelberg 1967; Ryzhov & Tamoikin 1970; Tsang & Kong 1981*b*; Stogryn 1983; Zhuck 1994; Michel & Lakhtakia 1995, 1996) and acoustics (Zhuck 1995, 1996), capable of handling strong CP fluctuations. This is because *perturbative schemes for averaging the renormalized equations operate with parameters that remain small even for strong CP fluctuations.*

Application of the strong-fluctuation theory, based on a renormalization approach, for elastodynamic homogenization appears to have gone unnoticed by earlier researchers. In order to remedy the situation, here we consider the homogenization of a macroscopically uniform elastic solid medium with fluctuating density and a stiffness tensor. The CP fluctuations are allowed to be strong, provided the fluctuations have a maximum length-scale less than the minimum wavelength of external stimuli.

2. Theoretical developments

(*a*) *Non-local effective constitutive operators*

Let all space be occupied by a solid whose stiffness tensor $c^{(\mathrm{r})}_{lmpq}$ and density $\rho^{(\mathrm{r})}$ are random functions of the spatial variable $\underline{x} = (x_1, x_2, x_3)$. After assuming an $\mathrm{e}^{-\mathrm{i}\omega t}$ time dependence† and using the standard summation convention for repeated indices, the equation of motion may be stated as

$$\partial_l t^{(\mathrm{r})}_{lm} + \omega^2 \rho^{(\mathrm{r})} u^{(\mathrm{r})}_m = -F_m, \tag{2.1}$$

where $t^{(\mathrm{r})}_{lm}$, $u^{(\mathrm{r})}_m$ and F_m denote the stress tensor, the displacement and the applied force, respectively, while the superscript '(r)' signifies random quantities. The strain tensor,

$$e^{(\mathrm{r})}_{pq} = \tfrac{1}{2}(\partial_p u^{(\mathrm{r})}_q + \partial_q u^{(\mathrm{r})}_p), \tag{2.2}$$

is related to the stress tensor via the stiffness tensor $c^{(\mathrm{r})}_{lmpq}$ in the constitutive relation,

$$t^{(\mathrm{r})}_{lm} = c^{(\mathrm{r})}_{lmpq} e^{(\mathrm{r})}_{pq}. \tag{2.3}$$

The usual symmetries

$$c^{(\mathrm{r})}_{lmpq} = c^{(\mathrm{r})}_{mlpq} = c^{(\mathrm{r})}_{lmqp} = c^{(\mathrm{r})}_{pqlm}$$

(Mal & Singh 1991) are assumed throughout this work.

In accordance with a suggestion made by Sabina & Willis (1988) in Appendix A of their paper, we define the following effective constitutive relations:

$$\langle c^{(\mathrm{r})}_{lmpq} e^{(\mathrm{r})}_{pq} \rangle \equiv c^{(\mathrm{e})}_{lmpq} \star \langle e^{(\mathrm{r})}_{pq} \rangle + \beta_{lmp} \star \langle u^{(\mathrm{r})}_p \rangle, \tag{2.4}$$

$$\langle \rho^{(\mathrm{r})} u^{(\mathrm{r})}_m \rangle \equiv \epsilon_{mpq} \star \langle e^{(\mathrm{r})}_{pq} \rangle + \rho^{(\mathrm{e})}_{mp} \star \langle u^{(\mathrm{r})}_p \rangle, \tag{2.5}$$

where the angular brackets indicate the expected values of the quantities enclosed. On taking the ensemble-average of (2.1) and (2.2) and making use of (2.3)–(2.5), we

† The symbol $\mathrm{i} = \sqrt{-1}$, except when $i = 1, 2, 3$ is used as an index.

get

$$(\partial_l c^{(\mathrm{e})}_{lmpq} + \omega^2 \epsilon_{mpq}) \star \langle e^{(\mathrm{r})}_{pq} \rangle + (\omega^2 \rho^{(\mathrm{e})}_{mp} + \partial_l \beta_{lmp}) \star \langle u^{(\mathrm{r})}_{p} \rangle = -F_m, \tag{2.6}$$

$$\langle e^{(\mathrm{r})}_{pq} \rangle = \tfrac{1}{2}[\partial_p \langle u^{(\mathrm{r})}_q \rangle + \partial_q \langle u^{(\mathrm{r})}_p \rangle] \tag{2.7}$$

as the mean field equations.

The nature of the operations defined on the right-hand sides of (2.4) and (2.5) makes the four deterministic effective constitutive operators (ECOs), $c^{(\mathrm{e})}_{lmpq}$, β_{lmp}, ϵ_{mpq} and $\rho^{(\mathrm{e})}_{mp}$, non-local; thus

$$\eta \star v \equiv \int \mathrm{d}^3 x' \, \eta(\underline{x}, \underline{x}') v(\underline{x}'), \tag{2.8}$$

where the ECOs have been denoted symbolically by η, and $v(\underline{x})$ is a testing function. Our focus lies here on a disordered medium with macroscopically uniform properties, i.e. the random functions $c^{(\mathrm{r})}_{lmpq}$ and $\rho^{(\mathrm{r})}$ are statistically homogeneous and homogeneously interrelated.† Then the integral on the right-hand side of (2.8) becomes a convolution integral as per

$$\eta \star v \equiv \int \mathrm{d}^3 x' \, \eta(\underline{x} - \underline{x}') v(\underline{x}'). \tag{2.9}$$

The shift invariance of the ECOs means that

$$\eta(\underline{x} - \underline{x}') = (2\pi)^{-3} \int \mathrm{d}^3 k \, \tilde{\eta}(\underline{k}) \exp[\mathrm{i}\underline{k} \cdot (\underline{x} - \underline{x}')]; \tag{2.10}$$

consequently,

$$\eta \star \exp(\mathrm{i}\underline{k} \cdot \underline{x}) \equiv \tilde{\eta}(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x}) \tag{2.11}$$

for an arbitrary wave vector $\underline{k} = (k_1, k_2, k_3)$. Equation (2.11) defines the spectral counterpart‡ $\tilde{\eta}(\underline{k})$ of an operator η with a shift-invariant kernel $\eta(\underline{x} - \underline{x}')$ for the remainder of this work.

When the body force has the spectral form

$$F_m(\underline{x}) = A_m(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x}), \tag{2.12}$$

$\underline{k}$ being a specified wave vector, the mean displacement and strain in a macroscopically uniform medium have, by virtue of (2.6) and (2.7), spectral forms as well:

$$\langle u^{(\mathrm{r})}_m(\underline{x}) \rangle = U_m(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x}), \tag{2.13}$$

$$\langle e^{(\mathrm{r})}_{pq}(\underline{x}) \rangle = E_{pq}(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x}). \tag{2.14}$$

† A random field is called statistically homogeneous in a narrow sense if its multipoint statistical moments of any order are shift-invariant functions of spatial variables. Let $\Upsilon^{(\mathrm{r})}_1(\underline{x})$ and $\Upsilon^{(\mathrm{r})}_2(\underline{x})$ each describe a statistically homogeneous field. These fields are said to be homogeneously interrelated in a narrow sense if their *mixed* multipoint statistical moments of any order are shift-invariant functions of spatial variables. In a wider sense, two random fields $\Upsilon^{(\mathrm{r})}_1(\underline{x})$ and $\Upsilon^{(\mathrm{r})}_2(\underline{x})$ are called statistically homogeneous and homogeneously interrelated if the aforementioned requirements are met by their first two statistical moments; that is, their average values $\langle \Upsilon^{(\mathrm{r})}_1(\underline{x}) \rangle$ and $\langle \Upsilon^{(\mathrm{r})}_2(\underline{x}) \rangle$ do not depend upon spatial variables, and their two-point second-order statistical moments,

$$\langle \Upsilon^{(\mathrm{r})}_1(\underline{x}) \Upsilon^{(\mathrm{r})}_1(\underline{x}') \rangle, \qquad \langle \Upsilon^{(\mathrm{r})}_2(\underline{x}) \Upsilon^{(\mathrm{r})}_2(\underline{x}') \rangle, \qquad \langle \Upsilon^{(\mathrm{r})}_1(\underline{x}) \Upsilon^{(\mathrm{r})}_2(\underline{x}') \rangle,$$

depend on $\underline{x}$ and $\underline{x}'$ through the difference variable $\underline{x} - \underline{x}'$ only.

‡ The spectral counterpart differs from the Fourier transform by a factor of $(2\pi)^3$.

Simultaneously, the non-local ECOs simplify so that for $v(\underline{x}) = V(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x})$ one has

$$\eta \star v \equiv \tilde{\eta}(\underline{k}) V(\underline{k}) \exp(\mathrm{i}\underline{k} \cdot \underline{x}), \qquad \eta = c^{(\mathrm{e})}_{lmpq}, \beta_{lmp}, \epsilon_{mpq}, \rho^{(\mathrm{e})}_{mp}, \tag{2.15}$$

with

$$\left.\begin{aligned} \lim_{\underline{k}\to\underline{0}} \tilde{c}^{(\mathrm{e})}_{lmpq}(\underline{k}) &= c^{\dagger}_{lmpq}, & \lim_{\underline{k}\to\underline{0}} \tilde{\beta}_{lmp}(\underline{k}) &= \beta^{\dagger}_{lmp}, \\ \lim_{\underline{k}\to\underline{0}} \tilde{\epsilon}_{mpq}(\underline{k}) &= \epsilon^{\dagger}_{mpq}, & \lim_{\underline{k}\to\underline{0}} \tilde{\rho}^{(\mathrm{e})}_{mp}(\underline{k}) &= \rho^{\dagger}_{mp} \end{aligned}\right\} \tag{2.16}$$

as the respective long-wavelength limits. As a non-local description must conform to local elasticity theory at a small wavenumber $k = \sqrt{\underline{k} \cdot \underline{k}}$ (Nowinski 1984), we must have

$$\beta^{\dagger}_{lmp} \equiv 0, \qquad \epsilon^{\dagger}_{mpq} \equiv 0, \qquad \rho^{\dagger}_{mp} \equiv \rho^{\dagger} \delta_{mp}, \tag{2.17}$$

with δ_{mp} as the Kronecker delta.

(*b*) *Renormalized equations*

Equations (2.1) and (2.2) do not easily yield to further analysis. Tractability is afforded by the definition of a homogeneous, anisotropic solid medium, with non-random density ρ and stiffness tensor c_{lmpq}, as a *comparison medium* (Hashin & Shtrikman 1962; Shermegor 1977; Willis 1980*a*, *b*; Talbot & Willis 1982*a*–*c*). Parenthetically, Appendix A provides a comparison of our approach with that pioneered by Willis: although the concept of a comparison medium is used in both approaches, they are also quite different from one another.

Suppose this anisotropic comparison medium (ACM) occupies all space and is driven by F_m. Then the displacement u_m in the ACM satisfies the following equation of motion:

$$c_{lmpq} \partial_l \partial_q u_p + \omega^2 \rho u_m = -F_m. \tag{2.18}$$

The solution of this linear equation can be written as (Wang & Achenbach 1995)

$$u_p = G_{pm} \star F_m, \tag{2.19}$$

using the Green operators G_{pm}, which are conveniently arranged in a 3×3 matrix Green operator $\hat{G} \equiv [G_{pm}]$. In the $\underline{k}$-space, the operator $\hat{G}$ converts to the spectral Green matrix

$$\hat{\tilde{G}}(\underline{k}) = [k^2 \hat{a}(\underline{k}) - \omega^2 \rho \hat{I}]^{-1}, \tag{2.20}$$

where $\hat{I}$ is the 3×3 identity matrix, and the 3×3 matrix $\hat{a}(\underline{k})$ has components

$$a_{mp}(\underline{k}) = \frac{k_l c_{lmpq} k_q}{k^2}. \tag{2.21}$$

The right-hand side of (2.20) can be simplified using matrix algebra. Thus,

$$\hat{\tilde{G}}(\underline{k}) = \frac{\hat{D}(\underline{k})}{\Delta(\underline{k})}, \tag{2.22}$$

where (Chen 1993, p. 14)

$$\hat{D}(\underline{k}) = k^4 \operatorname{adj} \hat{a}(\underline{k}) + \omega^2 \rho k^2 [\hat{a}(\underline{k}) - \hat{I} \operatorname{Tr} \hat{a}(\underline{k})] + \omega^4 \rho^2 \hat{I}, \tag{2.23}$$

$$\Delta(\underline{k}) = k^6 \det \hat{a}(\underline{k}) - \omega^2 \rho k^4 \operatorname{Tr}[\operatorname{adj} \hat{a}(\underline{k})] + \omega^4 \rho^2 k^2 \operatorname{Tr} \hat{a}(\underline{k}) - \omega^6 \rho^3. \tag{2.24}$$

Here and hereafter, adj, det and Tr, respectively, stand for the adjoint, the determinant and the trace of a matrix.

Let us now compare the ACM to our disordered medium. Substituting (2.2) and (2.3) in (2.1), we *synthesize* the following equation of motion:

$$c_{lmpq}\partial_l\partial_q u_p^{(\mathrm{r})} + \omega^2\rho u_m^{(\mathrm{r})} = -F_m - \delta F_m^{(\mathrm{r})}. \tag{2.25}$$

In this equation,

$$\delta F_m^{(\mathrm{r})} = \partial_l \delta c_{lmpq}^{(\mathrm{r})} e_{pq}^{(\mathrm{r})} + \omega^2 \delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}, \tag{2.26}$$

$$\delta c_{lmpq}^{(\mathrm{r})} = c_{lmpq}^{(\mathrm{r})} - c_{lmpq}, \tag{2.27}$$

$$\delta\rho^{(\mathrm{r})} = \rho^{(\mathrm{r})} - \rho, \tag{2.28}$$

are to be viewed as *perturbations* relative to the ACM, which becomes clear on comparing (2.18) and (2.25).

Equation (2.25) can be solved in the same way as (2.18) was, the solution being

$$u_j^{(\mathrm{r})} = u_j + \omega^2 G_{jm} \star [\delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}] + H_{jlm} \star [\delta c_{lmpq}^{(\mathrm{r})} e_{pq}^{(\mathrm{r})}]. \tag{2.29}$$

This solution, when substituted into (2.2), leads to

$$e_{ij}^{(\mathrm{r})} = e_{ij} + \omega^2 \mathcal{G}_{ijm} \star [\delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}] + \mathcal{H}_{ijlm} \star [\delta c_{lmpq}^{(\mathrm{r})} e_{pq}^{(\mathrm{r})}], \tag{2.30}$$

with $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ being the strain tensor in the ACM. In the two foregoing relations, we have employed the following three auxiliary operators:

$$H_{jlm} = \partial_l G_{jm}, \tag{2.31}$$

$$\mathcal{G}_{ijm} = \tfrac{1}{2}(\partial_i G_{jm} + \partial_j G_{im}), \tag{2.32}$$

$$\mathcal{H}_{ijlm} = \partial_l \mathcal{G}_{ijm}. \tag{2.33}$$

Now, the spectral Green matrix $\hat{\hat{G}}(\underline{k})$ can be advantageously recast as

$$\hat{\hat{G}}(\underline{k}) = \frac{\hat{b}(\underline{k})}{k^2} + \frac{\omega^2\rho}{k^2 \Delta(\underline{k})}\hat{b}(\underline{k}) \circ \hat{D}(\underline{k}), \tag{2.34}$$

where the symbol $\circ$ denotes matrix multiplication and $\hat{b}(\underline{k}) = \hat{a}^{-1}(\underline{k})$. Using (2.34), we find the spectral counterpart matrix of the operator $\mathcal{H}_{ijlm}$ as

$$\tilde{\mathcal{H}}_{ijlm}(\underline{k}) = -\frac{1}{2k^2}[k_j b_{im}(\underline{k}) + k_i b_{jm}(\underline{k})]k_l - \frac{\omega^2\rho}{2k^2\Delta(\underline{k})}[k_j b_{is}(\underline{k}) + k_i b_{js}(\underline{k})]k_l D_{sm}(\underline{k}), \tag{2.35}$$

where $b_{mn}(\underline{k})$ and $D_{mn}(\underline{k})$ are the respective elements of the matrices $\hat{b}(\underline{k})$ and $\hat{D}(\underline{k})$.

The first term on the right-hand side of (2.35) does not vanish as $k \to \infty$. Hence, $\mathcal{H}_{ijlm}$ can be split into two parts; thus in formal operator notation (Chigarev 1980; Zhuck 1994, 1996),

$$\mathcal{H}_{ijlm} = \mathcal{H}'_{ijlm} - S_{ijlm}, \tag{2.36}$$

where $\mathcal{H}'_{ijlm}$ is a singular integral operator associated with some infinitely small exclusion region and S_{ijlm} is a constant tensor. Both terms on the right-hand side of (2.36) are strongly dependent on an exclusion region, whereas their difference, being the left-hand side of (2.36), is defined uniquely.

Working, however, in the $\underline{k}$-space (i.e. the spectral domain), we follow Zhuck (1994, 1996) to postulate the spectral counterpart of $\mathcal{H}'_{ijlm}$ as

$$\tilde{\mathcal{H}}'_{ijlm}(\underline{k}) = \tilde{\mathcal{H}}_{ijlm}(\underline{k}) + S_{ijlm}, \tag{2.37}$$

where S_{ijlm} is an arbitrary renormalization tensor independent of $\underline{k}$. This definition relieves us of the necessity to decide on the shape and the size of the exclusion region; more importantly, the precise mathematical nature of $\mathcal{H}'_{ijlm}$ plays no role in the following analysis. However, we still have to eventually decide upon an optimally efficacious S_{ijlm}.

We are now in a position to renormalize (2.29) and (2.30). Let us define a new random second-rank tensor

$$f_{ij}^{(\mathrm{r})} = e_{ij}^{(\mathrm{r})} + S_{ijlm}\delta c_{lmpq}^{(\mathrm{r})} e_{pq}^{(\mathrm{r})}, \tag{2.38}$$

as well as a random fourth-rank tensor $\eta_{pqst}^{(\mathrm{r})}$ so that

$$e_{pq}^{(\mathrm{r})} = \eta_{pqst}^{(\mathrm{r})} f_{st}^{(\mathrm{r})}. \tag{2.39}$$

Then, (2.30) can be recast as

$$f_{ij}^{(\mathrm{r})} = e_{ij} + \omega^2 \mathcal{G}_{ijm} \star [\delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}] + \mathcal{H}'_{ijlm} \star [\xi_{lmpq}^{(\mathrm{r})} f_{pq}^{(\mathrm{r})}], \tag{2.40}$$

where

$$\xi_{lmpq}^{(\mathrm{r})} = \delta c_{lmst}^{(\mathrm{r})} \eta_{stpq}^{(\mathrm{r})} \tag{2.41}$$

plays the part of a random perturbation tensor associated with the CP fluctuations of the disordered medium. Similarly, (2.29) can be transformed into

$$u_j^{(\mathrm{r})} = u_j + \omega^2 G_{jm} \star [\delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}] + H_{jlm} \star [\xi_{lmpq}^{(\mathrm{r})} f_{pq}^{(\mathrm{r})}]. \tag{2.42}$$

Equations (2.40) and (2.42) constitute a set of renormalized equations. They contain the CPs ρ and c_{lmpq} of the ACM as well as the renormalization tensor S_{ijlm} as free parameters, whose optimal selection are key issues in our approach. Parenthetically, (2.40) and (2.42) respectively reduce to (2.30) and (2.29), if we set $S_{ijlm} \equiv 0$.

(*c*) *Effective perturbation operators*

Having obtained the renormalized equations (2.40) and (2.42), we proceed to establish linear relationships between the mean values of $\xi_{lmpq}^{(\mathrm{r})} f_{pq}^{(\mathrm{r})}$ and $\delta\rho^{(\mathrm{r})} u_m^{(\mathrm{r})}$ on the one hand and those of $f_{pq}^{(\mathrm{r})}$ and $u_p^{(\mathrm{r})}$ on the other.

In order to facilitate presentation, let us adopt an abbreviated formal notation:

$$\psi^{(\mathrm{r})} = \begin{bmatrix} f_{ij}^{(\mathrm{r})} \\ u_j^r \end{bmatrix}, \qquad \psi = \begin{bmatrix} e_{ij} \\ u_j \end{bmatrix}, \tag{2.43}$$

$$\Gamma = \begin{bmatrix} \mathcal{H}'_{ijlm} & \omega^2 \mathcal{G}_{ijm} \\ H_{jlm} & \omega^2 G_{jm} \end{bmatrix}, \qquad \Pi^{(\mathrm{r})} = \begin{bmatrix} \xi_{lmpq}^{(\mathrm{r})} & 0 \\ 0 & \delta\rho^{(\mathrm{r})} \end{bmatrix}. \tag{2.44}$$

Then, both (2.40) and (2.42) can be compactly stated as

$$\psi^{(\mathrm{r})} = \psi + \Gamma \Pi^{(\mathrm{r})} \star \psi^{(\mathrm{r})}, \tag{2.45}$$

whose *formal* solution is

$$\psi^{(\mathrm{r})} = (1 - \mathit{\Gamma}\mathit{\Pi}^{(\mathrm{r})})^{-1} \star \psi. \tag{2.46}$$

After using the consequent relations,

$$\langle \psi^{(\mathrm{r})} \rangle = \langle (1 - \mathit{\Gamma}\mathit{\Pi}^{(\mathrm{r})})^{-1} \rangle \star \psi, \tag{2.47}$$

$$\langle \mathit{\Pi}^{(\mathrm{r})}\psi^{(\mathrm{r})} \rangle = \langle \mathit{\Pi}^{(\mathrm{r})}(1 - \mathit{\Gamma}\mathit{\Pi}^{(\mathrm{r})})^{-1} \rangle \star \psi, \tag{2.48}$$

the deterministic operator

$$\mathit{\Pi}^{(\mathrm{e})} = \langle \mathit{\Pi}^{(\mathrm{r})}(1 - \mathit{\Gamma}\mathit{\Pi}^{(\mathrm{r})})^{-1} \rangle \langle (1 - \mathit{\Gamma}\mathit{\Pi}^{(\mathrm{r})})^{-1} \rangle^{-1} \tag{2.49}$$

can be shown to satisfy the identity

$$\langle \mathit{\Pi}^{(\mathrm{r})}\psi^{(\mathrm{r})} \rangle = \mathit{\Pi}^{(\mathrm{e})} \star \langle \psi^{(\mathrm{r})} \rangle. \tag{2.50}$$

Equivalently,

$$\langle \xi^{(\mathrm{r})}_{lmpq} f^{(\mathrm{r})}_{pq} \rangle = \breve{a}_{lmpq} \star \langle f^{(\mathrm{r})}_{pq} \rangle + \breve{b}_{lmp} \star \langle u^{(\mathrm{r})}_{p} \rangle, \tag{2.51}$$

$$\langle \delta\rho^{(\mathrm{r})} u^{(\mathrm{r})}_{m} \rangle = \breve{v}_{mpq} \star \langle f^{(\mathrm{r})}_{pq} \rangle + \breve{w}_{mp} \star \langle u^{(\mathrm{r})}_{p} \rangle, \tag{2.52}$$

where $\breve{a}_{lmpq}$, $\breve{b}_{lmp}$, $\breve{v}_{mpq}$ and $\breve{w}_{mp}$ are effective perturbation operators (EPOs) (Ryzhov & Tamoikin 1970; Zhuck 1996).

Suppose that the right-hand side of (2.49) has been ascertained somehow, either exactly or approximately, so that the EPOs are known. Now, from (2.4), (2.5), (2.38), (2.39) and (2.41), we have

$$\langle f^{(\mathrm{r})}_{ij} \rangle = \langle e^{(\mathrm{r})}_{ij} \rangle + S_{ijlm}(c^{(\mathrm{e})}_{lmpq} - c_{lmpq}) \star \langle e^{(\mathrm{r})}_{pq} \rangle + S_{ijlm}\beta_{lmp} \star \langle u^{(\mathrm{r})}_{p} \rangle, \tag{2.53}$$

$$\langle \xi^{(\mathrm{r})}_{lmpq} f^{(\mathrm{r})}_{pq} \rangle = (c^{(\mathrm{e})}_{lmpq} - c_{lmpq}) \star \langle e^{(\mathrm{r})}_{pq} \rangle + \beta_{lmp} \star \langle u^{(\mathrm{r})}_{p} \rangle, \tag{2.54}$$

$$\langle \delta\rho^{(\mathrm{r})} u^{(\mathrm{r})}_{m} \rangle = \epsilon_{mpq} \star \langle e^{(\mathrm{r})}_{pq} \rangle + (\rho^{(\mathrm{e})}_{mp} - \rho\delta_{mp}) \star \langle u^{(\mathrm{r})}_{p} \rangle. \tag{2.55}$$

On comparing (2.54) and (2.55) with (2.51) and (2.52), and taking (2.53) into account, the ECOs and the EPOs can be related as follows:

$$c^{(\mathrm{e})}_{lmpq} - c_{lmpq} = \breve{a}_{lmpq} + \breve{a}_{lmrs} S_{rstu}(c^{(\mathrm{e})}_{tupq} - c_{tupq}), \tag{2.56}$$

$$\beta_{lmp} = \breve{b}_{lmp} + \breve{a}_{lmrs} S_{rstu} \beta_{tup}, \tag{2.57}$$

$$\epsilon_{mpq} = \breve{v}_{mpq} + \breve{v}_{mrs} S_{rstu}(c^{(\mathrm{e})}_{tupq} - c_{tupq}), \tag{2.58}$$

$$\rho^{(\mathrm{e})}_{mp} = \rho\delta_{mp} + \breve{w}_{mp} + \breve{v}_{mrs} S_{rstu} \beta_{tup}. \tag{2.59}$$

The operator equations (2.56)–(2.59) reduce to algebraic relations in the $\underline{k}$-space, whence $\tilde{c}^{(\mathrm{e})}_{lmpq}(\underline{k})$, $\tilde{\beta}_{lmp}(\underline{k})$, $\tilde{\epsilon}_{mpq}(\underline{k})$, $\tilde{\rho}^{(\mathrm{e})}_{mp}(\underline{k})$ can be easily determined from $\tilde{a}_{lmpq}(\underline{k})$, $\tilde{b}_{lmp}(\underline{k})$, $\tilde{v}_{mpq}(\underline{k})$ and $\tilde{w}_{mp}(\underline{k})$. Thus, the crux of the matter is the evaluation of the right-hand side of (2.49). But that cannot be accomplished rigorously and accurately. Some approximations are needed for further analysis, which we discuss in § 2 *d*.

In passing, however, let us remark on a disordered medium with uniform density. We can then choose $\rho = \rho^{(\mathrm{r})}$ so that $\delta\rho^{(\mathrm{r})} = 0$ and the reductions $\breve{b}_{lmp} \equiv 0$, $\breve{v}_{mpq} \equiv 0$ and $\breve{w}_{mp} \equiv 0$ follow. Hence, $\beta_{lmp} \equiv 0$, $\epsilon_{mpq} \equiv 0$, and $\rho^{(\mathrm{e})}_{mp} \equiv \rho\delta_{mp}$, so that the density remains unaltered during homogenization.

(*d*) *Bilocal approximation*

Following an argument laid out by Finkelberg (1967), let us expand the right-hand side of (2.49) in powers of $\Pi^{(\mathrm{r})}$; thus,

$$\Pi^{(\mathrm{e})} = \sum_{n=1}^{\infty} \Pi_n, \tag{2.60}$$

where

$$\Pi_1 = \langle \Pi^{(\mathrm{r})} \rangle, \tag{2.61}$$

$$\Pi_n = \langle \Pi^{(\mathrm{r})} (\Gamma \Pi^{(\mathrm{r})})^{n-1} \rangle - \sum_{m=1}^{n-1} \Pi_m \langle (\Gamma \Pi^{(\mathrm{r})})^{n-m} \rangle; \quad n = 2, 3, 4, \ldots. \tag{2.62}$$

The bilocal approximation (Lifshitz & Rosenzveig 1946) is obtained after retaining only the first two terms in (2.60), i.e.

$$\Pi^{(\mathrm{e})} \approx \langle \Pi^{(\mathrm{r})} \rangle + \langle \Pi^{(\mathrm{r})} \Gamma \Pi^{(\mathrm{r})} \rangle - \langle \Pi^{(\mathrm{r})} \rangle \Gamma \langle \Pi^{(\mathrm{r})} \rangle, \tag{2.63}$$

and the bilocally approximated EPOs are easily ascertained therefrom. After transforming to the $\underline{k}$-space, (2.56)–(2.59) can be solved.

In particular, on taking the long-wavelength limit $k \to 0$, we get

$$\begin{aligned} c^{\dagger}_{lmpq} \approx{}& c_{lmpq} + \langle \xi^{(\mathrm{r})}_{lmpq} \rangle + \langle \xi^{(\mathrm{r})}_{lmrs} \rangle S_{rstu} \langle \xi^{(\mathrm{r})}_{tupq} \rangle \\ &- \tfrac{1}{2} \omega^2 \rho \int \mathrm{d}^3 k' \, \frac{k'_t}{k'^2} B^{lmrs}_{tupq}(\underline{k}') \tilde{G}_{vu}(\underline{k}') [k'_s b_{rv}(\underline{k}') + k'_r b_{sv}(\underline{k}')] \\ &+ \int \mathrm{d}^3 k' \, B^{lmrs}_{tupq}(\underline{k}') \left\{ S_{rstu} - \frac{k'_t}{2k'^2} [k'_s b_{ru}(\underline{k}') + k'_r b_{su}(\underline{k}')] \right\}, \end{aligned} \tag{2.64}$$

$$\beta^{\dagger}_{lmp} \approx -\tfrac{1}{2} \mathrm{i} \omega^2 \int \mathrm{d}^3 k' \, B_{lmst}(\underline{k}') [k'_s \tilde{G}_{tp}(\underline{k}') + k'_t \tilde{G}_{sp}(\underline{k}')], \tag{2.65}$$

$$\epsilon^{\dagger}_{mpq} \approx -\mathrm{i} \int \mathrm{d}^3 k' \, k'_s B_{stpq}(-\underline{k}') \tilde{G}_{mt}(\underline{k}'), \tag{2.66}$$

$$\rho^{\dagger}_{mp} \approx [\rho + \langle \delta\rho^{(\mathrm{r})} \rangle] \delta_{mp} + \omega^2 \int \mathrm{d}^3 k' \, B(\underline{k}') \tilde{G}_{mp}(\underline{k}'). \tag{2.67}$$

In these expressions, the spectral densities $B^{lmrs}_{tupq}(\underline{k})$, $B_{lmst}(\underline{k})$ and $B(\underline{k})$ are related to the correlation functions

$$C^{lmrs}_{tupq}(\underline{x} - \underline{x}') = \langle \xi^{(\mathrm{r})}_{lmrs}(\underline{x}) \xi^{(\mathrm{r})}_{tupq}(\underline{x}') \rangle - \langle \xi^{(\mathrm{r})}_{lmrs}(\underline{x}) \rangle \langle \xi^{(\mathrm{r})}_{tupq}(\underline{x}') \rangle, \tag{2.68}$$

$$C_{lmst}(\underline{x} - \underline{x}') = \langle \xi^{(\mathrm{r})}_{lmst}(\underline{x}) \delta\rho^{(\mathrm{r})}(\underline{x}') \rangle - \langle \xi^{(\mathrm{r})}_{lmst}(\underline{x}) \rangle \langle \delta\rho^{(\mathrm{r})}(\underline{x}') \rangle, \tag{2.69}$$

$$C(\underline{x} - \underline{x}') = \langle \delta\rho^{(\mathrm{r})}(\underline{x}) \delta\rho^{(\mathrm{r})}(\underline{x}') \rangle - \langle \delta\rho^{(\mathrm{r})}(\underline{x}) \rangle \langle \delta\rho^{(\mathrm{r})}(\underline{x}') \rangle \tag{2.70}$$

through the Fourier transform; thus,

$$B(\underline{k}) = (2\pi)^{-3} \int \mathrm{d}^3 x \, C(\underline{x} - \underline{x}') \exp[-\mathrm{i} \underline{k} \cdot (\underline{x} - \underline{x}')], \tag{2.71}$$

etc. We must point out that the poles of $\hat{\tilde{G}}(\underline{k}')$, which coincide with the zeros of $\Delta(\underline{k}')$, are supposed to occur at complex locations. If the ACM is non-dissipative, then it has to be made slightly dissipative as an artifice to evaluate the $\underline{k}'$-integrals.

In order to simplify further analysis, we assume that $C_{lmst}(\underline{x}-\underline{x}')$ are even functions of $\underline{x}-\underline{x}'$, which means that $B_{lmst}(\underline{k}')$ are odd functions of $\underline{k}'$. Because of the similar parities of $\tilde{G}_{tp}(\underline{k}')$ and $\tilde{G}_{sp}(\underline{k}')$, the integrand in (2.65) is an odd function of $\underline{k}'$. Hence, integration within infinite limits on the right-hand side of (2.65) yields $\beta^{\dagger}_{lmp}=0$ in the bilocal approximation. For the same reason, $\epsilon^{\dagger}_{mpq}=0$ also; and the HDM turns out to be adequately described by an effective stiffness tensor $c^{\dagger}_{lmpq}$ and an effective density tensor $\rho^{\dagger}_{mp}$.

(*e*) *Strong CP fluctuations*

The right-hand sides of (2.64) and (2.67) can be further simplified by eliminating secular terms (Tsang & Kong 1981*b*). The secular terms in these expressions contain at least one mean value $\langle\delta\rho^{(\mathrm{r})}\rangle$ or $\langle\xi^{(\mathrm{r})}_{lmpq}\rangle$ as a multiplier.

To help visualize the notion of secularity, let us introduce positive deterministic constants σ_ρ and σ_c such that $\delta\rho^{(\mathrm{r})}\sim\sigma_\rho$, $\delta c^{(\mathrm{r})}_{lmpq}\sim\sigma_c$ and $\xi^{(\mathrm{r})}_{lmpq}\sim\sigma_c$. Let L_ρ and L_c be the characteristic length-scales of the random density and stiffness fluctuations, respectively. A distinctive feature of a secular term is that its growth with the strength of property fluctuations, i.e. σ_ρ and σ_c, cannot be tamed through small parameters, e.g. L_ρ, L_c and ω (Tsang & Kong 1981*b*).

If $\langle\delta\rho^{(\mathrm{r})}\rangle$ and $\langle\xi^{(\mathrm{r})}_{lmpq}\rangle$ are not null-valued, then any solution found by truncating the series (2.60) contains secular terms. We must therefore require the two conditions

$$\langle\delta\rho^{(\mathrm{r})}\rangle=0, \tag{2.72}$$

$$\langle\xi^{(\mathrm{r})}_{lmpq}\rangle=0, \tag{2.73}$$

to hold. In turn, these conditions imply that we must set

$$\rho=\langle\rho^{(\mathrm{r})}\rangle, \tag{2.74}$$

and choose c_{lmpq} and S_{ijlm} such that

$$\langle(c^{(\mathrm{r})}_{lmst}-c_{lmst})\{[1+S(c^{(\mathrm{r})}-c)]^{-1}\}_{stpq}\rangle=0. \tag{2.75}$$

In an effort to satisfy the requirement (2.75), let us concentrate on the last term on the right-hand side of (2.64). This term, being proportional to σ_c^2, is secular. To eliminate this term, let us choose the elements of the renormalization tensor as solutions of the following set of coupled equations:

$$S_{rstu}\int \mathrm{d}^3k\, B^{lmrs}_{tupq}(\underline{k})=\int \mathrm{d}^3k\,\frac{k_t}{2k^2}B^{lmrs}_{tupq}(\underline{k})[k_s b_{ru}(\underline{k})+k_r b_{su}(\underline{k})]. \tag{2.76}$$

Consequently, (2.72)–(2.75) yield

$$c^{\dagger}_{lmpq}\approx c_{lmpq}-\tfrac{1}{2}\omega^2\rho\int \mathrm{d}^3k\,\frac{k_t}{k^2}B^{lmrs}_{tupq}(\underline{k})\tilde{G}_{vu}(\underline{k})[k_s b_{rv}(\underline{k})+k_r b_{sv}(\underline{k})], \tag{2.77}$$

$$\rho^{\dagger}_{mp}\approx\rho\delta_{mp}+\omega^2\int \mathrm{d}^3k\, B(\underline{k})\tilde{G}_{mp}(\underline{k}). \tag{2.78}$$

In order to ascertain the regimes of validity of the expressions (2.77) and (2.78), we have to compare the second terms with respect to the first terms on the right-hand sides of these two equations. Let us use the slightly less accurate but more convenient expressions

$$c^{\dagger}_{lmpq} \approx c_{lmpq} - \tfrac{1}{2}\omega^2\rho \int \mathrm{d}^3k\, \frac{k_t}{k^4} B^{lmrs}_{tupq}(\underline{k}) b_{vu}(\underline{k})[k_s b_{rv}(\underline{k}) + k_r b_{sv}(\underline{k})], \tag{2.79}$$

$$\rho^{\dagger}_{mp} \approx \rho\delta_{mp} + \omega^2 \int \mathrm{d}^3k\, \frac{1}{k^2} B(\underline{k}) b_{mp}(\underline{k}), \tag{2.80}$$

which emerge after evaluating the spectral Green matrix elements encountered in the integrands of (2.77) and (2.78) at $\omega = 0$. We find that the results (2.77) and (2.78) are adequate, provided the two criteria,

$$(\omega L_c)^2 \left(\frac{\sigma_c}{c^{(t)}}\right)^2 \frac{\rho}{c^{(t)}} \ll 1, \tag{2.81}$$

$$(\omega L_\rho)^2 \left(\frac{\sigma_\rho}{\rho}\right)^2 \frac{\rho}{c^{(t)}} \ll 1, \tag{2.82}$$

hold, with $c^{(t)}$ as the magnitude of a typical value of c_{lmpq}. The most attractive feature of these criteria is that they authorize the application of (2.77) and (2.78) when the CP fluctuations are strong (i.e. $\sigma_c/c^{(t)} \gg 1$ and/or $\sigma_\rho/\rho \gg 1$), provided the length-scales L_c and L_ρ are small and/or the angular frequency ω is low.

By retracing the preceding analysis, we can show that the factor $\omega^2\sigma_\rho^2 L_\rho^2/c^{(t)}\rho$ controlling the applicability of (2.78) is, in fact, unaffected by the choice of the renormalization tensor S_{ijlm}. In contrast, a proper choice of S_{ijlm} turns out to be of crucial importance when applying (2.77). Indeed, a departure from the S_{ijlm} prescribed by (2.76) brings in secular terms, making the right-hand side of (2.64) applicable only for weak CP fluctuations.

(*f*) *Optimal choice of the renormalization tensor*

Depending on the statistical properties of the random stiffness tensor $c^{(\mathrm{r})}_{lmpq}$, the renormalization tensor S_{ijlm} may be ascertained analytically. Let us consider a random perturbation tensor $\xi^{(\mathrm{r})}_{lmpq}$ whose correlation functions (2.68) are of the following form:

$$C^{lmrs}_{tupq}(\underline{x} - \underline{x}') = \int \mathrm{d}^3a\, w(\underline{a}) F^{lmrs}_{tupq}(\theta_{\underline{a}}). \tag{2.83}$$

Here, $\underline{a} = (a_1, a_2, a_3)$ is real-valued; the arbitrary functions $F^{lmrs}_{tupq}(\theta)$ are specified on ellipsoidal iso-correlation surfaces

$$\theta^2_{\underline{a}} = \frac{(x_1 - x_1')^2}{a_1^2} + \frac{(x_2 - x_2)^2}{a_2^2} + \frac{(x_3 - x_3)^2}{a_3^2}; \tag{2.84}$$

and the weight function $w(\underline{a})$ is also arbitrary except that it must be non-zero only in the first octant of the $\underline{a}$-space and obeys the normalization condition $\int \mathrm{d}^3a\, w(\underline{a}) = 1$. As an example, $w(\underline{a}) = \delta(\underline{a} - \underline{L})$, where $\underline{L} = (L_1, L_2, L_3)$ is a vector of length-scales and $\delta(\underline{x})$ is the Dirac delta function.

A straightforward calculation reveals that

$$B^{lmrs}_{tupq}(\underline{k}) = \int \mathrm{d}^3a\, w(\underline{a}) a_1 a_2 a_3 \varPhi^{lmrs}_{tupq}(\tau_{\underline{a}}), \tag{2.85}$$

where

$$\Phi^{lmrs}_{tupq}(\tau_{\underline{a}}) = \frac{1}{2\pi^2 \tau_{\underline{a}}} \int_0^{+\infty} \mathrm{d}\theta\, \theta \sin(\tau_{\underline{a}}\theta) F^{lmrs}_{tupq}(\theta) \tag{2.86}$$

depend on

$$\tau_{\underline{a}} = [(k_1 a_1)^2 + (k_2 a_2)^2 + (k_3 a_3)^2]^{1/2}. \tag{2.87}$$

Inserting (2.85) into (2.76), interchanging the order of the $\underline{k}$-integration and the $\underline{a}$-integration, and normalizing k_1, k_2 and k_3 into τ_1, τ_2 and τ_3 via

$$k_1 = \frac{\tau_1}{a_1}, \qquad k_2 = \frac{\tau_2}{a_2}, \qquad k_3 = \frac{\tau_3}{a_3}, \tag{2.88}$$

we obtain Zhuck (1994, 1996)

$$\int_0^{+\infty} \tau^2 \Phi^{lmrs}_{tupq}(\tau)\, \mathrm{d}\tau \left[4\pi S_{rstu} - \int \mathrm{d}^3 a\, w(\underline{a}) \int_{\Sigma} \mathrm{d}\Sigma\, \Psi(\underline{\tau}) \right] = 0. \tag{2.89}$$

Here

$$\Psi(\underline{\tau}) = \tfrac{1}{2} k^{-2} k_t [b_{ru}(\underline{k}) k_s + b_{su}(\underline{k}) k_r], \tag{2.90}$$

with k_1, k_2 and k_3 being expressed in terms of τ_1, τ_2 and τ_3 via (2.88); $\Sigma \equiv \{\tau_1^2 + \tau_2^2 + \tau_3^2 = 1\}$ is a sphere of unit radius centred at the origin of the $\underline{\tau}$-space; and $\mathrm{d}\Sigma$ is a surface element of sphere Σ. Equation (2.89) is satisfied by

$$S_{rstu} = \frac{1}{4\pi} \int \mathrm{d}^3 a\, w(\underline{a}) \int_{\Sigma} \mathrm{d}\Sigma\, \Psi(\underline{\tau}), \tag{2.91}$$

so that, after reverting to the original variables k_1, k_2 and k_3, we get

$$S_{rstu} = \frac{1}{8\pi} \int \mathrm{d}^3 a\, w(\underline{a}) a_1 a_2 a_3 \int_{\mathcal{S}_{\underline{a}}} \mathrm{d}S\, \frac{k_t [k_s b_{ru}(\underline{k}) + k_r b_{su}(\underline{k})]}{k^2 [k_1^2 a_1^4 + k_2^2 a_2^4 + k_3^2 a_3^4]^{1/2}}, \tag{2.92}$$

where $\mathrm{d}S$ is a surface element of the ellipsoid

$$\mathcal{S}_{\underline{a}} \equiv \{(k_1 a_1)^2 + (k_2 a_2)^2 + (k_3 a_3)^2 = 1\} \tag{2.93}$$

specified in the $\underline{k}$-space as a function of $\underline{a}$.

Another meaningful example is provided by the random perturbation tensor ξ_{lmpq} whose correlation functions (2.68) depend only on the distance $R = |\underline{x} - \underline{x}'|$ but not on $\underline{x}$ and $\underline{x}'$ separately; i.e.

$$C^{lmrs}_{tupq}(\underline{x} - \underline{x}') \equiv C^{lmrs}_{tupq}(R), \tag{2.94}$$

and, consequently,

$$B^{lmrs}_{tupq}(\underline{k}) \equiv B^{lmrs}_{tupq}(k) = \frac{1}{2\pi^2 k} \int_0^{+\infty} \mathrm{d}R\, R \sin kR C^{lmrs}_{tupq}(R). \tag{2.95}$$

These correlation functions describe, among other things, a particulate composite medium prepared by randomly dispersing spherical inclusions in a uniform matrix medium. Using (2.95) in (2.76), we get

$$S_{rstu} = \frac{1}{8\pi} \int_{4\pi} \mathrm{d}\Omega_{\underline{k}}\, \frac{k_t}{k^2} [k_s b_{ru}(\underline{k}) + k_r b_{su}(\underline{k})], \tag{2.96}$$

where $\mathrm{d}\Omega_{\underline{k}}$ stands for the solid-angle element in $\underline{k}$-space.

In continuation of this example, let the ACM be isotropic, i.e.

$$c_{lmpq} = \lambda\delta_{lm}\delta_{pq} + \mu(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{2.97}$$

with Lamé constants λ and μ. Then,

$$b_{ru}(\underline{k}) = \frac{1}{\mu}\left(\delta_{ru} - \frac{k_r k_u}{k^2}\frac{\lambda+\mu}{\lambda+2\mu}\right), \tag{2.98}$$

and (2.96) yields the renormalization tensor

$$S_{rstu} = \frac{(3\lambda+8\mu)(\delta_{rt}\delta_{su} + \delta_{ru}\delta_{st}) - 2(\lambda+\mu)\delta_{rs}\delta_{tu}}{30\mu(\lambda+2\mu)}. \tag{2.99}$$

3. Application to a particulate composite medium

(*a*) *Spherical inclusions in an isotropic matrix medium*

Let us apply the developed strong-fluctuation approach for homogenizing a particulate composite medium prepared by randomly dispersing identical, uniform, anisotropic, spherical inclusions in an isotropic matrix medium having a density $\rho^{(1)}$ and Lamé constants $\lambda^{(1)}$ and $\mu^{(1)}$. The density of each inclusion is $\rho^{(2)}$ and its stiffness tensor in its own crystallographic (i.e. material) frame of reference is denoted by $c^{(\mathrm{c})}_{lmpq}$. Perfect bonding between the spherical inclusions and the matrix medium is assumed.

Let the random density of this disordered medium be denoted as

$$\rho^{(\mathrm{r})}(\underline{x}) = \theta^{(\mathrm{r})}(\underline{x})\rho^{(2)} + [1 - \theta^{(\mathrm{r})}(\underline{x})]\rho^{(1)}, \tag{3.1}$$

where the characteristic random function $\theta^{(\mathrm{r})}(\underline{x}) = 1$ when $\underline{x}$ is occupied by the inclusion medium, and $\theta^{(\mathrm{r})}(\underline{x}) = 0$ otherwise. The inclusions are located at random positions such that the inclusion volume fraction

$$\langle\theta^{(\mathrm{r})}(\underline{x})\rangle = v_2 \tag{3.2}$$

is constant over the entire space, and the two-inclusion correlation function

$$\langle\theta^{(\mathrm{r})}(\underline{x})\theta^{(\mathrm{r})}(\underline{x}')\rangle = p(R) \tag{3.3}$$

is isotropic. The statistical topology of the chosen particulate composite is further delineated by a non-dimensional correlation function (Shermegor 1977, ch. 4; Lin *et al*. 1994)

$$f(R) = \frac{p(R) - v_2^2}{v_2(1-v_2)}, \tag{3.4}$$

whose spatial Fourier transform $\varphi(k)$ is defined through (2.95). We note for the sake of completeness that $\varphi(k)$ is a real non-negative function, being the spectral density of the correlation function of a real-valued random field $[\theta^{(\mathrm{r})}(\underline{x}) - v_2]\sqrt{v_2(1-v_2)}$ (Gihman *et al*. 1979).

The random stiffness tensor of the disordered medium is given by the following expressions:

$$c^{(\mathrm{r})}_{lmpq}(\underline{x}) = \lambda^{(1)}\delta_{lm}\delta_{pq} + \mu^{(1)}(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{3.5}$$

if $\underline{x}$ is occupied by the matrix medium; and

$$c^{(\mathrm{r})}_{lmpq}(\underline{x}) = [\alpha^{(\mathrm{r})}_{li}\alpha^{(\mathrm{r})}_{mj}\alpha^{(\mathrm{r})}_{pk}\alpha^{(\mathrm{r})}_{qn}]c^{(\mathrm{c})}_{ijkn}, \tag{3.6}$$

if otherwise. Here, α_{li} is the cosine of an angle between the lth axis of the laboratory frame of reference and the ith axis of the crystallographic frame of reference of a particular inclusion. Although the inclusions are identical, they differ from each other in the orientation of their crystallographic axes. The different orientations are characterized by the products $\alpha^{(\mathrm{r})}_{li}\alpha^{(\mathrm{r})}_{mj}\alpha^{(\mathrm{r})}_{pk}\alpha^{(\mathrm{r})}_{qn}$ in (3.6). We assume that all orientations are equally probable, and that the orientation of a particular inclusion is statistically independent of the position of this inclusion as well as of the orientation and position of any other inclusion.

(*b*) *Isotropic comparison medium*

Clearly, the HDM is isotropic. Therefore, let us choose the ACM to be isotropic, with its stiffness tensor given by (2.97). In view of the spherical symmetry of the correlation function (3.4), the correlation functions (2.68) of the random perturbation tensor must satisfy (2.94). Hence, the renormalization tensor S_{rstu} can be calculated from (2.99). The resulting determination of S_{rstu} is invariant with respect to the transformations from the crystallographic frame of reference of any particular inclusion and the laboratory frame; i.e.

$$S_{rstu} = S^{(\mathrm{c})}_{rstu}. \tag{3.7}$$

The random perturbation tensor of (2.41) is then given by

$$\xi^{(\mathrm{r})}_{lmpq}(\underline{x}) = [\kappa^{(1)} - \tfrac{2}{3}\zeta^{(1)}]\delta_{lm}\delta_{pq} + \zeta^{(1)}(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}) \equiv \xi^{(1)}_{lmpq} \tag{3.8}$$

when $\underline{x}$ is occupied by the matrix medium, with

$$\kappa^{(1)} = \frac{(\lambda + 2\mu)[3(\lambda^{(1)} - \lambda) + 2(\mu^{(1)} - \mu)]}{4\mu + 3\lambda^{(1)} + 2\mu^{(1)}}, \tag{3.9}$$

$$\zeta^{(1)} = \frac{15\mu(\lambda + 2\mu)(\mu^{(1)} - \mu)}{\mu(9\lambda + 14\mu) + 2\mu^{(1)}(3\lambda + 8\mu)}. \tag{3.10}$$

In order to calculate $\xi^{(\mathrm{r})}_{lmpq}(\underline{x})$ when $\underline{x}$ is occupied by the inclusion medium, let us employ the well-known rule

$$\xi^{(\mathrm{r})}_{lmpq}(\underline{x}) = \alpha^{(\mathrm{r})}_{li}(\underline{x})\alpha^{(\mathrm{r})}_{mj}(\underline{x})\alpha^{(\mathrm{r})}_{pk}(\underline{x})\alpha^{(\mathrm{r})}_{qn}(\underline{x})\xi^{(\mathrm{c})}_{ijkn}(\underline{x}). \tag{3.11}$$

Although $\alpha^{(\mathrm{r})}_{li}(\underline{x})$, etc., are random due to random orientations of the inclusions, the tensor

$$\xi^{(\mathrm{c})}_{lmpq} \equiv \delta c^{(\mathrm{c})}_{lmst}\eta^{(\mathrm{c})}_{stpq} \tag{3.12}$$

is deterministic. Here

$$\delta c^{(\mathrm{c})}_{lmst} = [c^{(\mathrm{c})}_{lmst} - \lambda\delta_{lm}\delta_{st} - \mu(\delta_{ls}\delta_{mt} + \delta_{lt}\delta_{ms})], \tag{3.13}$$

and the tensor $\eta^{(\mathrm{c})}_{rspq}$ is defined to ensure the inversion, in a manner analogous to that of (2.39), of the equation

$$e^{(r,c)}_{pq} = \eta^{(\mathrm{c})}_{pqst}f^{(r,c)}_{st}, \tag{3.14}$$

where $e_{pq}^{(r,c)}$ are the random strain components in the crystallographic frame and

$$f_{ij}^{(r,c)} = e_{ij}^{(r,c)} + S_{ijlm}\delta c_{lmpq}^{(\mathrm{c})} e_{pq}^{(r,c)}, \tag{3.15}$$

is the crystallographic-frame counterpart of (2.38).

Thus, on taking the ensemble-average of $\xi_{lmpq}^{(\mathrm{r})}(\underline{x})$, we obtain the following expression:

$$\langle \xi_{lmpq}^{(\mathrm{r})}(\underline{x}) \rangle = [v_1\kappa^{(1)} + v_2\kappa^{(2)} - \tfrac{2}{3}(v_1\zeta^{(1)} + v_2\zeta^{(2)})]\delta_{lm}\delta_{pq} \\ + (v_1\zeta^{(1)} + v_2\zeta^{(2)})(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{3.16}$$

where $v_1 = 1 - v_2$, while

$$\kappa^{(2)} = \tfrac{1}{9}\xi_{rrss}^{(\mathrm{c})}, \tag{3.17}$$

$$\zeta^{(2)} = \tfrac{1}{10}(\xi_{rsrs}^{(\mathrm{c})} - \tfrac{1}{3}\xi_{rrss}^{(\mathrm{c})}). \tag{3.18}$$

Making use of (3.16) in (2.73), we get a system of two coupled equations,

$$v_1\kappa^{(1)} + v_2\kappa^{(2)} = 0, \tag{3.19}$$

$$v_1\zeta^{(1)} + v_2\zeta^{(2)} = 0, \tag{3.20}$$

to determine the Lamé constants λ and μ of the isotropic comparison medium (ICM). The density ρ of the ICM turns out to be

$$\rho = v_1\rho^{(1)} + v_2\rho^{(2)} \tag{3.21}$$

from (2.74).

(*c*) *Effective constitutive properties*

For the chosen disordered medium, the correlation functions defined in (2.68)–(2.70) can be written in terms of $f(R)$ as (Shermegor 1977, ch. 4),

$$C_{tupq}^{lmrs}(\underline{x} - \underline{x}') = v_1 v_2 f(R) D_{tupq}^{lmrs}, \tag{3.22}$$

$$C_{lmst}(\underline{x} - \underline{x}') = v_1 v_2 f(R) D_{lmst}, \tag{3.23}$$

$$C(\underline{x} - \underline{x}') = v_1 v_2 f(R) D. \tag{3.24}$$

In these relations,

$$D_{tupq}^{lmrs} = \xi_{lmrs}^{(1)}\xi_{tupq}^{(1)} + \langle \alpha_{ld}^{(\mathrm{r})}\alpha_{me}^{(\mathrm{r})}\alpha_{rf}^{(\mathrm{r})}\alpha_{sg}^{(\mathrm{r})}\alpha_{th}^{(\mathrm{r})}\alpha_{ui}^{(\mathrm{r})}\alpha_{pj}^{(\mathrm{r})}\alpha_{qk}^{(\mathrm{r})}\rangle_{or}\xi_{defg}^{(\mathrm{c})}\xi_{hijk}^{(\mathrm{c})} \\ - \langle \alpha_{ld}^{(\mathrm{r})}\alpha_{me}^{(\mathrm{r})}\alpha_{rf}^{(\mathrm{r})}\alpha_{sg}^{(\mathrm{r})}\rangle_{or}\xi_{defg}^{(\mathrm{c})}\xi_{tupq}^{(1)} \\ - \langle \alpha_{th}^{(\mathrm{r})}\alpha_{ui}^{(\mathrm{r})}\alpha_{pj}^{(\mathrm{r})}\alpha_{qk}^{(\mathrm{r})}\rangle_{or}\xi_{lmrs}^{(1)}\xi_{hijk}^{(\mathrm{c})}, \tag{3.25}$$

$$D_{lmst} = [\rho^{(1)} - \rho^{(2)}][\xi_{lmst}^{(1)} - \langle \alpha_{li}^{(\mathrm{r})}\alpha_{mj}^{(\mathrm{r})}\alpha_{sk}^{(\mathrm{r})}\alpha_{tn}^{(\mathrm{r})}\rangle_{or}\xi_{ijkn}^{(\mathrm{c})}], \tag{3.26}$$

$$D = [\rho^{(1)} - \rho^{(2)}]^2, \tag{3.27}$$

where $\langle \dots \rangle_{or}$ denotes the average over all orientations of the crystallographic frame of reference relative to the laboratory frame of reference. We observe that $C_{lmst}(\underline{x} - \underline{x}')$ in (3.23) is an even function of $\underline{x} - \underline{x}'$, thereby leading to $\beta_{lmp}^{\dagger} = 0$ and $\epsilon_{mpq}^{\dagger} = 0$, as argued in § 2 *d*.

Having in mind that the HDM is expected to be isotropic, we remark that the second term on the right-hand side of (2.77) must turn out to actually have a very simple structure. In other words, (2.77) has to simplify to

$$c^{\dagger}_{lmpq} = c_{lmpq} + a^{\dagger}_{lmpq}, \tag{3.28}$$

where

$$a^{\dagger}_{lmpq} = \delta\lambda\delta_{lm}\delta_{pq} + \delta\mu(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{3.29}$$

$$\delta\lambda = \tfrac{1}{15}(2a^{\dagger}_{llpp} - a^{\dagger}_{lmlm}), \tag{3.30}$$

$$\delta\mu = \tfrac{1}{10}(a^{\dagger}_{lmlm} - \tfrac{1}{3}a^{\dagger}_{llpp}). \tag{3.31}$$

The right-hand sides of (3.30) and (3.31) can be evaluated by using the spectral Green matrix of the ICM, i.e. by substituting

$$\tilde{G}_{mj}(\underline{k}) = \frac{\delta_{mj}}{\Delta^{(\mathrm{s})}(k)} - \frac{(\lambda+\mu)k_m k_j}{\Delta^{(\mathrm{s})}(k)\Delta^{(\mathrm{l})}(k)} \tag{3.32}$$

on the right-hand side of (2.77) with

$$\Delta^{(\mathrm{s})}(k) = \mu k^2 - \omega^2\rho, \tag{3.33}$$

$$\Delta^{(\mathrm{l})}(k) = (\lambda+2\mu)k^2 - \omega^2\rho. \tag{3.34}$$

Thus,

$$\delta\lambda = -\tfrac{4}{3}\pi\frac{\omega^2\rho}{\mu}v_1v_2\int_0^{+\infty}\mathrm{d}k\,\varphi(k)k^2\left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{\omega^2\rho-(\lambda+3\mu)k^2}{\Delta^{(\mathrm{s})}(k)\Delta^{(\mathrm{l})}(k)}L + \frac{M}{\Delta^{(\mathrm{s})}(k)}\right], \tag{3.35}$$

$$\delta\mu = -\tfrac{4}{3}\pi\frac{\omega^2\rho}{\mu}v_1v_2\int_0^{+\infty}\mathrm{d}k\,\varphi(k)k^2\left[\frac{\lambda+\mu}{\lambda+2\mu}\frac{\omega^2\rho-(\lambda+3\mu)k^2}{\Delta^{(\mathrm{s})}(k)\Delta^{(\mathrm{l})}(k)}N + \frac{P}{\Delta^{(\mathrm{s})}(k)}\right], \tag{3.36}$$

where the constants L, M, N and P are expressible† through contractions of D^{lmrs}_{tupq}. The effective stiffness tensor of the chosen HDM can thus be estimated as

$$c^{\dagger}_{lmpq} = \lambda^{\dagger}\delta_{lm}\delta_{pq} + \mu^{\dagger}(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}), \tag{3.37}$$

where

$$\lambda^{\dagger} = \lambda + \delta\lambda, \tag{3.38}$$

$$\mu^{\dagger} = \mu + \delta\mu \tag{3.39}$$

are the effective Lamé constants of the HDM.

The calculation of the effective density tensor $\rho^{\dagger}_{mp}$ is also accomplished by substituting the spectral Green matrix (3.32) in (2.78) and taking advantage of spherical symmetry of the relevant spectral density $B(k) = v_1v_2\varphi(k)D$. Then the second term on the right-hand side of (2.78) differs from the second-rank unit tensor δ_{mp} only by the scalar factor

$$\delta\rho = \tfrac{4}{3}\pi\omega^2 v_1v_2 D\int_0^{+\infty}\mathrm{d}k\,\varphi(k)k^2\frac{[(2\lambda+5\mu)k^2 - 3\omega^2\rho]}{\Delta^{(\mathrm{s})}(k)\Delta^{(\mathrm{l})}(k)}. \tag{3.40}$$

Therefore, $\rho^{\dagger}_{mp} = \rho^{\dagger}\delta_{mp}$, where

$$\rho^{\dagger} = \rho + \delta\rho \tag{3.41}$$

is the effective density of the HDM.

† The detailed forms of these four constants are available in Appendix B for anisotropic spherical inclusions, and the particular case of spherical inclusions made of a solid with cubic crystallographic symmetry is covered in § 3 *e*.

(*d*) *Dissipation in the HDM*

Let us now consider the effect of attenuation of the mean field propagating in an originally non-dissipative disordered medium. A physical interpretation of this attenuation lies in the transference of part of the energy from the coherent to the incoherent displacements as a result of scattering by randomly dispersed and randomly oriented inclusions.

Mathematically, the dissipation can manifest itself by making the ECPs $\lambda^\dagger$, $\mu^\dagger$ and $\rho^\dagger$ complex-valued. By virtue of (2.74), (2.75) and (2.99), let us choose the ICM to be non-dissipative (i.e. characterized by real positive λ, μ and ρ) when the disordered medium is itself non-dissipative. We therefore focus on calculating the real and imaginary parts of $\delta\lambda$, $\delta\mu$ and $\delta\rho$ appearing in (3.38), (3.39) and (3.41).

As the chosen ICM is non-dissipative, the integrands in (3.35), (3.36) and (3.40) have two simple poles on the Re[k]-axis in the complex k-plane. These poles are $k = k^{(\mathrm{l})}$ and $k = k^{(\mathrm{s})}$, where

$$k^{(\mathrm{l})} = \omega\sqrt{\frac{\rho}{\lambda + 2\mu}}, \tag{3.42}$$

$$k^{(\mathrm{s})} = \omega\sqrt{\frac{\rho}{\mu}} \tag{3.43}$$

are, respectively, the longitudinal and the shear wavenumbers in the ICM. After displacing the poles artificially into the upper half of the complex k-plane, using the Cauchy residue theorem, and setting the imaginary parts of the poles to zero again, the integrals in (3.35), (3.36) and (3.40) can be evaluated for ascertaining the complex-valued $\delta\lambda$, $\delta\mu$ and $\delta\rho$ in closed form. Expressions for Re[$\delta\lambda$], Re[$\delta\mu$], and Re[$\delta\rho$], respectively, emerge on taking the integrals on the right-hand sides of (3.35), (3.36) and (3.40) in the principal value sense at the poles $k = k^{(\mathrm{l})}$ and $k = k^{(\mathrm{s})}$; however, these expressions are not relevant in the present context. More importantly, we get

$$\mathrm{Im}[\delta\lambda] = -\tfrac{2}{3}\pi^2 v_1 v_2 \omega^3 \rho^{3/2}\left[\frac{L}{(\lambda+2\mu)^{5/2}} + \frac{M-L}{\mu^{5/2}}\right]\lim_{k\to 0}\varphi(k), \tag{3.44}$$

$$\mathrm{Im}[\delta\mu] = -\tfrac{2}{3}\pi^2 v_1 v_2 \omega^3 \rho^{3/2}\left[\frac{N}{(\lambda+2\mu)^{5/2}} + \frac{P-N}{\mu^{5/2}}\right]\lim_{k\to 0}\varphi(k), \tag{3.45}$$

$$\mathrm{Im}[\delta\rho] = \tfrac{2}{3}\pi^2 v_1 v_2 \omega^3 \rho^{1/2} D\left[\frac{1}{(\lambda+2\mu)^{3/2}} + \frac{2}{\mu^{3/2}}\right]\lim_{k\to 0}\varphi(k). \tag{3.46}$$

Next, the effective wavenumbers in the HDM are given by

$$k_e^{(\mathrm{l})} = \omega\sqrt{\frac{\rho^\dagger}{\lambda^\dagger + 2\mu^\dagger}}, \tag{3.47}$$

$$k_e^{(\mathrm{s})} = \omega\sqrt{\frac{\rho^\dagger}{\mu^\dagger}} \tag{3.48}$$

for longitudinal and shear waves, respectively. From (3.38), (3.39) and (3.41), and correct to the first order in $\delta\lambda$, $\delta\mu$ and $\delta\rho$, we get

$$k_e^{(\mathrm{l})} \approx k^{(\mathrm{l})}(1 + \gamma^{(\mathrm{l})}), \tag{3.49}$$

$$k_e^{(\mathrm{s})} \approx k^{(\mathrm{s})}(1 + \gamma^{(\mathrm{s})}), \tag{3.50}$$

where

$$\gamma^{(l)} = \tfrac{1}{2}\left(\frac{\delta\rho}{\rho} - \frac{\delta\lambda + 2\delta\mu}{\lambda + 2\mu}\right), \tag{3.51}$$

$$\gamma^{(s)} = \tfrac{1}{2}\left(\frac{\delta\rho}{\rho} - \frac{\delta\mu}{\mu}\right). \tag{3.52}$$

Then, in accordance with (3.44)–(3.46), we immediately obtain

$$\begin{aligned}\mathrm{Im}[\gamma^{(l)}] = \tfrac{1}{3}\pi^2 v_1 v_2 \omega^3 \rho^{3/2} \Bigg\{ \frac{D}{\rho^2}\left[\frac{1}{(\lambda+2\mu)^{3/2}} + \frac{2}{\mu^{3/2}}\right] \\ + \frac{1}{\lambda+2\mu}\left[\frac{L+2N}{(\lambda+2\mu)^{5/2}} + \frac{M-L+2(P-N)}{\mu^{5/2}}\right]\Bigg\} \lim_{k\to 0} \varphi(k),\end{aligned} \tag{3.53}$$

$$\begin{aligned}\mathrm{Im}[\gamma^{(s)}] = \tfrac{1}{3}\pi^2 v_1 v_2 \omega^3 \rho^{3/2} \Bigg\{ \frac{D}{\rho^2}\left[\frac{1}{(\lambda+2\mu)^{3/2}} + \frac{2}{\mu^{3/2}}\right] \\ + \frac{1}{\mu}\left[\frac{N}{(\lambda+2\mu)^{5/2}} + \frac{P-N}{\mu^{5/2}}\right]\Bigg\} \lim_{k\to 0} \varphi(k).\end{aligned} \tag{3.54}$$

If $\mathrm{Im}[\gamma^{(l)}] > 0$ and $\mathrm{Im}[\gamma^{(s)}] > 0$, the mean elastodynamic fields in the HDM attenuate. The attenuation is guaranteed for inclusions with cubic crystallographic symmetry (Auld 1990), as will be shown next, but remains unproved for more general anisotropic spherical inclusions, the inclusion as well as the matrix media being non-dissipative.

(*e*) *Inclusion medium with cubic crystallographic symmetry*

The stiffness tensor of an elastic solid with cubic crystallographic symmetry contains only three independent elements (Auld 1990). Suppose the inclusions in the disordered medium of interest are made of a medium with cubic symmetry; then,

$$c^{(c)}_{lmpq} = \lambda^{(c)}\delta_{lm}\delta_{pq} + \mu^{(c)}(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}) + \nu^{(c)}\sum_{n=1}^{3}\delta_{ln}\delta_{mn}\delta_{pn}\delta_{qn}. \tag{3.55}$$

Accordingly, from (3.12) the perturbation tensor turns out to be

$$\begin{aligned}\xi^{(c)}_{lmpq} = (\kappa^{(c)} - \varepsilon^{(c)})\delta_{lm}\delta_{pq} \\ + \zeta^{(c)}(\delta_{lp}\delta_{mq} + \delta_{lq}\delta_{mp}) + (3\varepsilon^{(c)} - 2\zeta^{(c)})\sum_{n=1}^{3}\delta_{ln}\delta_{mn}\delta_{pn}\delta_{qn},\end{aligned} \tag{3.56}$$

where

$$\kappa^{(c)} = \frac{(\lambda+2\mu)[3(\lambda^{(c)} - \lambda) + 2(\mu^{(c)} - \mu) + \nu^{(c)}]}{4\mu + 3\lambda^{(c)} + 2\mu^{(c)} + \nu^{(c)}}, \tag{3.57}$$

$$\zeta^{(c)} = \frac{15\mu(\lambda+2\mu)(\mu^{(c)} - \mu)}{\mu(9\lambda + 14\mu) + 2\mu^{(c)}(3\lambda + 8\mu)}, \tag{3.58}$$

$$\varepsilon^{(c)} = \frac{5\mu(\lambda+2\mu)[2(\mu^{(c)} - \mu) + \nu^{(c)}]}{\mu(9\lambda + 14\mu) + (2\mu^{(c)} + \nu^{(c)})(3\lambda + 8\mu)}. \tag{3.59}$$

Hence, (3.17) and (3.18) yield $\kappa^{(2)} = \kappa^{(c)}$, $\zeta^{(2)} = (3/5)(\varepsilon^{(c)} + \zeta^{(c)})$, so that (3.19) and (3.20) simplify to

$$v_1\kappa^{(1)} + v_2\kappa^{(c)} = 0, \tag{3.60}$$

$$5v_1\zeta^{(1)} + 3v_2(\varepsilon^{(c)} + \zeta^{(c)}) = 0, \tag{3.61}$$

respectively, from which equations the Lamé constants of the ICM can be determined.

Application of (B 1)–(B 4) from Appendix B yields

$$L = 3(\kappa^{(1)} - \kappa^{(c)})^2 - 4Q, \tag{3.62}$$

$$M = 3(\kappa^{(1)} - \kappa^{(c)})^2 - 10Q, \tag{3.63}$$

$$N = 6Q, \tag{3.64}$$

$$P = 15Q, \tag{3.65}$$

where

$$Q = \tfrac{1}{25}\{2[\zeta^{(c)} - \zeta^{(1)}]^2 + 3[\varepsilon^{(c)} - \tfrac{2}{3}\zeta^{(1)}]^2\} > 0. \tag{3.66}$$

Because $L + 2N > 0$, $M - L + 2(P - N) > 0$, $N > 0$ and $P - N > 0$, attenuation of the mean elastodynamic fields is signified for this case by virtue of (3.49), (3.50), (3.53) and (3.54).

(*f*) *Comparison with available elastostatic results*

Let us now compare the results of the presented strong-fluctuation approach with results available for certain simple disordered media. Specifically, we consider only the elastostatic ECPs of an HDM with spherical inclusions in an isotropic matrix medium.

On taking the limit $\omega \to 0$, we see from (3.35), (3.36) and (3.40) that $\delta\lambda$, $\delta\mu$ and $\delta\rho$ are proportional to ω^2 in the long-wavelength limit. Simultaneously, the ECPs $\lambda^\dagger$, $\mu^\dagger$ and $\rho^\dagger$, respectively, approach the CPs λ, μ and ρ of the comparison medium, by virtue of (3.38), (3.39) and (3.41). Clearly then, λ, μ and ρ delineate the elastostatic response characteristics of the HDM.

Next, an elastostatic homogenization theory based on a self-consistent T-matrix scheme was proposed by Middya *et al.* (1985) for a two-phase material of cubic crystallographic symmetry. When one specializes equations (22a) and (22b) of Middya *et al.* (1985) to the case of an isotropic host material, they coincide exactly with our (3.60) and (3.61), respectively, leading to identical estimates of the ECPs.

Let us further assume that the inclusion medium is isotropic, i.e. $\nu^{(c)} = 0$. Then, (3.60) and (3.61) reproduce the well-known results of Hill (1963), Budiansky (1965) and others (e.g. Berryman 1980; see also Norris (1985), who rederived them in another form). It is useful to render yet another form of these classical results,

$$K = v_1K^{(1)} + v_2K^{(2)} - 3v_1v_2\frac{(K^{(1)} - K^{(2)})^2}{4\mu + 3(v_1K^{(2)} + v_2K^{(1)})}, \tag{3.67}$$

$$\mu = v_1\mu^{(1)} + v_2\mu^{(2)} + \frac{6(K + 2\mu)(\mu^{(1)} - \mu)(\mu^{(2)} - \mu)}{5\mu(3K + \mu)}, \tag{3.68}$$

in terms of the bulk modulus $K = \lambda + \frac{2}{3}\mu$ and shear modulus μ of the ICM, with $K^{(1)} = \lambda^{(1)} + \frac{2}{3}\mu^{(1)}$, $K^{(2)} = \lambda^{(c)} + \frac{2}{3}\mu^{(c)}$ and $\mu^{(2)} = \mu^{(c)}$. Equation (3.67) conforms to equation (2.1) of Chigarev (1980).† This form of (3.67) and (3.68) seems especially suitable for the development of a numerical solution for K and μ via an iterative procedure, with the first two terms on the right of both equations constituting the initial guess necessary to start the chosen iterative procedure.

Finally, when the shear moduli of the inclusion and the matrix media coincide as well, we conclude from (3.68) that $\mu = \mu^{(1)} (= \mu^{(2)})$, and equation (3.67) turns into a well-known result of Hill (1963). As emphasized by Middya *et al*. (1985), any reasonable homogenization exercise must restore Hill's result when the shear modulus of the disordered medium is uniform.

4. Concluding remarks

Disordered media abound. One way to understand their response characteristics is to homogenize them. We have presented a renormalization approach that holds well even for strong CP fluctuations, provided the characteristic length-scales of the fluctuations are small in relation to the minimum wavelength of the external stimuli. Our approach has been shown to handle arbitrary anisotropy of the constitutive and the statistical properties.

Several immediate extensions of our work are possible. First, textural anisotropy due to preferred orientation of anisotropic and/or non-spherical inclusions can be studied. Inclusions of several different species dispersed together in a matrix medium can be accommodated through proper choices of correlation functions. Second, better estimates of the ECPs are possible: for instance, by using a nonlinear approximation, which has the merit of being energetically consistent with the Bethe–Salpeter equation for the second moments of the displacement under the ladder approximation (Rytov *et al*. 1987). Finally, second-order statistical moments of a random elastic wave field could be examined, e.g. by applying the methodology developed by Brekhovskikh (1985) and Stogryn (1990) in electromagnetic field theory.

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Appendix A. Comparison of the presented renormalization formulation with the White formulation

In response to a suggestion made by a reviewer, we provide here a comparison of our approach with that pioneered by Willis (1980*a*, *b*). Both approaches employ the concept of a comparison medium, which is a key element in all approaches that view a composite medium as a combination of a comparison medium and a perturbation. Respectively, these two enter formulations through (i) the Green functions, and (ii) the equivalent sources induced within the perturbation regions by the incident field.

The use of volume integral equations can be traced to the classic papers of Lifshitz & Rosenzveig (1946) and Lifshitz & Parkhomovski (1948) referring, respectively, to elastostatics and elastodynamics of composite media with uniform mass density. A

† An erroneous plus sign in front of the second term on the right-hand side of Chigarev's equation should be replaced by a minus sign.

general system of coupled integral equations for composite media with mismatches in both the stiffness tensor and the mass density was developed by Willis (1980*a*, *b*); cf. eqns (2.24) and (2.25) of Willis (1980*a*).

The integral equation formulation proposed by us, the manner in which it has been applied to handle composite media, as well as the physical significance of the comparison medium used by us, are very different from the formulation developed by Willis and colleagues. The reasons for this assertion are as follows.

1. The unknown quantities in the Willis formulation are the momentum polarization π and the stress polarization τ, or, equivalently, the displacement u and the strain e, relative to the comparison medium; see eqn (2.14) of Willis (1980*a*). In our system of renormalized integral equations (2.40) and (2.42), the unknowns are the random displacement $u_j^{(\mathrm{r})}$ and a random field variable $f_{ij}^{(\mathrm{r})}$. The latter is artificial, but it can be related to the strain via (2.38).

2. The kernel S_x in the Willis formulation is essentially a generalized function of $\underline{x}-\underline{x}'$, since it possesses a strong delta-function-like singularity when the source and the field points coincide (i.e. $\underline{x} = \underline{x}'$). This singularity can be ascertained by treating the second mixed derivative with respect to spatial variables in eqn (2.26) of Willis (1980*a*) with the help of generalized function theory. The kernel S_x differs by a simple scale factor from the Green function $\mathcal{H}_{ijlm}$ defined by us through (2.33). In our renormalized integral equations (2.40) and (2.42), the kernel with the strongest singularity is $\mathcal{H}'_{ijlm}$. This kernel, obtained by removing the delta-function singularity from $\mathcal{H}_{ijlm}$ ($\sim S_x$ of Willis) in accordance with the prescription (2.36), exhibits a lower degree of singularity than $\mathcal{H}_{ijlm}$ and S_x. This means that the kernels in our integral equations are less singular than those in the Willis formulation. Actually, the two formulations belong to different mathematical classes: the equations of Willis are, in fact, integro-differential ones, whereas ours are singular integral equations.

3. As far as the determination of ECPs is concerned, the Willis formulation has been applied to discrete particulate composite media only. Various versions of multiple-scattering theory for particulate composites have been proposed using the quasi-crystalline approximation (Willis 1980*b*), the stochastic variational principle (Talbot & Willis 1982*a*–*c*), and a self-consistent method combined with a Galerkin-type approximation for single-scattering problems (Sabina & Willis 1988; Sabina *et al.* 1993; Smyshlyaev *et al.* 1993). In contrast, any composite medium, whether continuously random or discretely random, is regarded as a fluctuating continuum in our formulation. The ECPs of the fluctuating continuum are calculated with the help of integral equations, like those of Dyson, for the mean values of random displacement $u^{(\mathrm{r})}$ and that of an *artificial* field $f^{(\mathrm{r})}$. The kernels of these equations, as well as the ECPs, are given as expansions in powers of a small parameter characteristic of the strong-fluctuation approach. Thus, our continuum approach differs from the multiple-scattering approach of Willis and colleagues.

4. Finally, as regards the ACM, it was assumed to coincide with the host medium by Willis (1980*b*), chosen to produce the Hashin–Shtrikman lower and upper bounds for the ECPs (Talbot & Willis 1982*a*, *b*), or identified with the HDM

(Talbot & Willis 1982*a*, *b*; Sabina & Willis 1988; Sabina *et al*. 1993; Smyshlyaev *et al*. 1993). Our ACM, however, is chosen to meet the requirements (2.74) and (2.75) so as eliminate secular terms from an asymptotic solution for the ECPs and thus make that solution applicable when the CP fluctuations are strong. Indeed, by construction, the CPs of our ACM differ from the ECPs by a scattering contribution $\sim \omega^2$. Hence, at non-zero frequencies, our ACM does not coincide with the HDM and thus cannot be the same as the comparison ($\equiv$ effective) material of self-consistent formulations.

Clearly, the strong-fluctuation formulation presented by us does not belong to the class of self-consistent formulations. It should be viewed rather as yielding a perturbative scheme for averaging the renormalized equations that employ specific perturbative parameters that remain small even for strong CP fluctuations.

Appendix B. Auxiliary parameters for effective Lamé constants in § 3 *c*

Expressions for the four constants L, M, N and P appearing in (3.35) and (3.36) are as follows:

$$\begin{aligned} L = 3\kappa^{(1)2} - \tfrac{8}{15}\zeta^{(1)2} + \tfrac{1}{75}[8\zeta^{(1)}\xi^{(c)}_{lmlm} + 2(\tfrac{2}{3}\zeta^{(1)} - 25\kappa^{(1)})\xi^{(c)}_{llrr} \\ + 2\xi^{(c)}_{llrr}\xi^{(c)}_{ttpp} + 4\xi^{(c)}_{llrs}\xi^{(c)}_{rspp} - \xi^{(c)}_{lmrr}\xi^{(c)}_{ttlm} - 2\xi^{(c)}_{lmrs}\xi^{(c)}_{rslm}], \end{aligned} \tag{B 1}$$

$$\begin{aligned} M = 3\kappa^{(1)2} - \tfrac{4}{3}\zeta^{(1)2} + \tfrac{1}{15}[4\zeta^{(1)}\xi^{(c)}_{lmlm} - 2(5\kappa^{(1)} + \tfrac{2}{3}\zeta^{(1)})\xi^{(c)}_{llrr} \\ + 2\xi^{(c)}_{llrs}\xi^{(c)}_{rspp} - \xi^{(c)}_{lmrs}\xi^{(c)}_{rslm}], \end{aligned} \tag{B 2}$$

$$\begin{aligned} N = \tfrac{4}{5}\zeta^{(1)2} + \tfrac{1}{50}[8\zeta^{(1)}(\tfrac{1}{3}\xi^{(c)}_{llrr} - \xi^{(c)}_{lmlm}) \\ + \xi^{(c)}_{lmrr}\xi^{(c)}_{ttlm} + 2\xi^{(c)}_{lmrs}\xi^{(c)}_{rslm} - \tfrac{1}{3}(\xi^{(c)}_{llrr}\xi^{(c)}_{ttpp} + 2\xi^{(c)}_{llrs}\xi^{(c)}_{rspp})], \end{aligned} \tag{B 3}$$

$$P = 2\zeta^{(1)2} + \tfrac{1}{30}[4\zeta^{(1)}(\ \xi^{(c)}_{llrr} - 3\xi^{(c)}_{lmlm}) + 3\xi^{(c)}_{lmrs}\xi^{(c)}_{rslm} - \xi^{(c)}_{llrs}\xi^{(c)}_{rspp}]. \tag{B 4}$$

The right-hand sides of these equations contain invariants of a fourth-rank tensor $\xi^{(c)}_{lmpq}$ and an eighth-rank tensor $\xi^{(c)}_{lmrs}\xi^{(c)}_{tupq}$.

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