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Restricted equivalence of paired epsilon–negative and mu–negative layers to a negative phase–velocity material (*alias* left–handed material)

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Abstract: The time–harmonic electromagnetic responses of (a) a bilayer made of an epsilon–negative layer and a mu–negative layer, and (b) a single layer of a negative phase–velocity material are compared. Provided all layers are electrically thin, a restricted equivalence between (a) and (b) exists. The restricted equivalence depends on the linear polarization state and the transverse wavenumber. Implications for perfect lenses and parallel–plate waveguides are considered.

Key words: Negative real permittivity – negative real permeability – negative phase velocity – parallel–plate waveguide – perfect lens – phase velocity – Poynting vector

1. Introduction

This communication is inspired by the ongoing spate of papers published on the inappropriately designated *left-handed materials* which are macroscopically homogeneous and display negative phase velocities in relation to the time–averaged Poynting vector, but are not chiral [1]. Nominally, such a material is deemed to possess a relative permittivity scalar $\epsilon_r = \epsilon_r' + i\epsilon_r''$ and a relative permeability scalar $\mu_r = \mu_r' + i\mu_r''$, both dependent on the angular frequency ω, such that both $\epsilon_r' < 0$ and $\mu_r' < 0$ in some spectral regime[1]. Originally conceived by Veselago [3], these materials have been artificially realized quite recently [4, 5]. Their fascinating electromagnetic properties can have technological implications of massive proportions [6], but those implications remain speculative at this time. Although these materials have been variously named [1], we prefer the name *negative phase–velocity* (NPV) materials as the least ambiguous of all extant names.

Using transmission–line analysis and lumped parameters, Alu and Engheta [7] have recently suggested a new route to realizing NPV materials: Take a thin layer of an *epsilon–negative* (EN) material: it has a negative real permittivity scalar but a positive real permeability scalar. Stick it to a thin layer of a *mu–negative* (MN) material, which has a negative real permeability scalar and a positive real permittivity scalar. Provided the two layers are sufficiently thin, the paired EN–MN layers *could* function effectively as a NPV material. The clear attraction of this scheme is that EN and MN layers are easier to manufacture, very likely, than the NPV materials fabricated thus far [4, 5].

Our objective here is to examine the suggested scheme using continuum field theory, and to establish a restricted equivalence of an EN–MN bilayer to a NPV material. Implications for parallel–plate waveguides [7] and perfect lenses [8] are deduced therefrom.

A note about notation: Vectors are in boldface, column vectors are boldface and enclosed in square brackets, while matrixes are denoted by Gothic letters; ϵ_0 and μ_0 are the free–space permittivity and permeability, respectively; and $k_0 = \omega(\epsilon_0\mu_0)^{1/2}$ is the free–space wavenumber. A cartesian coordinate system is used, with $\boldsymbol{u}_x$, $\boldsymbol{u}_y$ and $\boldsymbol{u}_z$ as the unit vectors.

2. Bilayer theory in brief

Consider the layers $0 < z < d_a$ and $d_a < z < d_a + d_b$. Their constitutive relations are as follows:

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{r}) &= \epsilon_0\epsilon_a\,\boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{B}(\boldsymbol{r}) &= \mu_0\mu_a\,\boldsymbol{H}(\boldsymbol{r}) \end{aligned}\right\}, \qquad 0 < z < d_a\,, \tag{1}$$

$$\left.\begin{aligned} \boldsymbol{D}(\boldsymbol{r}) &= \epsilon_0\epsilon_b\,\boldsymbol{E}(\boldsymbol{r}) \\ \boldsymbol{B}(\boldsymbol{r}) &= \mu_0\mu_b\,\boldsymbol{H}(\boldsymbol{r}) \end{aligned}\right\}, \qquad d_a < z < d_a + d_b\,. \tag{2}$$

[1] The condition for the phase velocity and the time-averaged Poynting vector to be oppositely directed is $(|\varepsilon_r| - \varepsilon_r')\,(|\mu_r| - \mu_r') > \varepsilon_r''\mu_r''$, which permits – more generally – ε_r' and/or μ_r' to be negative [2]. An exp $(-i\omega t)$ time-dependence having been assumed here, $\varepsilon_r'' > 0$ and $\mu_r'' > 0$ at all $\omega > 0$ for all passive materials.

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The constitutive parameters present in the foregoing equations are complex–valued with positive imaginary parts (as befits any passive medium). The two half–spaces on either sides of the bilayer are vacuous.

Without loss of generality, the electromagnetic field phasors everywhere can be written as [9]

$$\left.\begin{aligned} \boldsymbol{E}(\boldsymbol{r}) = \tilde{\boldsymbol{e}}(z)\, e^{i\kappa x} \\ \boldsymbol{H}(\boldsymbol{r}) = \tilde{\boldsymbol{h}}(z)\, e^{i\kappa x} \end{aligned}\right\}, \quad -\infty < z < \infty\,, \tag{3}$$

where the transverse wavenumber $\kappa \in [0, \infty)$. The fields inside the bilayer must follow the 4×4 matrix ordinary differential equation [9, 10]

$$\frac{d}{dz}\,[\boldsymbol{f}(z)] = i\mathcal{P}(z)\,[\boldsymbol{f}(z)]\,, \quad 0 < z < d_a + d_b\,. \tag{4}$$

In this equation, $[\boldsymbol{f}(z)] = \mathrm{col}\,[\tilde{e}_x(z),\, \tilde{e}_y(z),\, \tilde{h}_x(z),\, \tilde{h}_y(z)]$ is a column vector, while the 4×4 matrix function $\mathcal{P}(z)$ is piecewise uniform as

$$\mathcal{P}(z) = \begin{cases} \mathcal{P}_a\,, & 0 < z < d_a \\ \mathcal{P}_b\,, & d_a < z < d_a + d_b \end{cases}\,, \tag{5}$$

where

$$\mathcal{P}_{a,b} = \begin{bmatrix} 0 & 0 & 0 & -\dfrac{\kappa^2}{\omega\epsilon_0\epsilon_{a,b}} + \omega\mu_0\mu_{a,b} \\ 0 & 0 & -\omega\mu_0\mu_{a,b} & 0 \\ 0 & -\omega\epsilon_0\epsilon_{a,b} + \dfrac{\kappa^2}{\omega\mu_0\mu_{a,b}} & 0 & 0 \\ \omega\epsilon_0\epsilon_{a,b} & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

The only nonzero elements of the matrixes $\mathcal{P}_{a,b}$ appear on their antidiagonals, of which the $(2,3)$ and the $(3,2)$ elements are relevant to *s*-polarized fields, and the $(1,4)$ and the $(4,1)$ elements to the *p*-polarized fields.

The solution of (4) is straightforward, because the matrix $\mathcal{P}$(z) is piecewise uniform [12]. Thus, the algebraic relation

$$[\boldsymbol{f}(d_b + d_a)] = e^{i\mathcal{P}_b d_b}\, e^{i\mathcal{P}_a d_a}\, [\boldsymbol{f}(0)] \tag{7}$$

is sufficient to solve both reflection/transmission problems as well as guided–wave propagation problems. The two matrix exponentials on the right side of (7) cannot be interchanged – unless the matrixes $\mathcal{P}_a$ and $\mathcal{P}_b$ also commute, which is possible with dissimilar materials *only* in quite special circumstances [10, 11].

3. Analysis

Matrixes $\mathcal{P}_{a,b}$ have $\pm\sqrt{k_0^2\epsilon_{a,b}\mu_{a,b} - \kappa^2} = \pm\alpha_{a,b}$ as their eigenvalues. Provided that $|\alpha_{a,b}|\, d_{a,b} \ll 1$ (i.e., both layers are electrically thin), the approximations

$$e^{i\mathcal{P}_{a,b}d_{a,b}} \simeq \mathcal{I} + i\mathcal{P}_{a,b}d_{a,b} \tag{8}$$

can be made, with $\mathcal{I}$ as the 4×4 identity matrix. Then

$$e^{i\mathcal{P}_b d_b}\, e^{i\mathcal{P}_a d_a} \simeq \mathcal{I} + i\mathcal{P}_a d_a + i\mathcal{P}_b d_b \simeq e^{i\mathcal{P}_a d_a}\, e^{i\mathcal{P}_b d_b}\,, \tag{9}$$

and the two layers in the bilayer can be interchanged without significant effect [13].

Let us now consider a single layer of relative permittivity ϵ_{eq}, relative permeability μ_{eq} and thickness d_{eq}. Quantities $\mathcal{P}_{eq}$ and α_{eq} can be defined in analogy to $\mathcal{P}_a$ and α_a. Two thickness ratios are defined as

$$p_{a,b} = \frac{d_{a,b}}{d_{eq}} \geq 0\,, \tag{10}$$

in order to compare the single layer with the previously described bilayer. There is no hidden restriction on the non–negative real numbers p_a and p_b.

Provided that $|\alpha_{eq}|\, d_{eq} \ll 1$ (i.e., the single layer is electrically thin as well), the approximation

$$e^{i\mathcal{P}_{eq}d_{eq}} \simeq \mathcal{I} + i\mathcal{P}_{eq}d_{eq} \tag{11}$$

can be made. Equations (9) and (11) permit us to establish the following equivalences between a bilayer and a single layer:

(i) *s–polarization:* The only nonzero field components are E_y, H_x and H_z. Therefore, the equality of the $(2,3)$ elements of $p_a\mathcal{P}_a + p_b\mathcal{P}_b$ and $\mathcal{P}_{eq}$, and likewise of the $(3,2)$ elements, has to be guaranteed for equivalence; thus, the equations

$$p_a\mu_a + p_b\mu_b = \mu_{eq}\,, \tag{12}$$

$$p_a\epsilon_a + p_b\epsilon_b - \left(\frac{\kappa}{k_0}\right)^2 \left(\frac{p_a}{\mu_a} + \frac{p_b}{\mu_b}\right) = \epsilon_{eq} - \left(\frac{\kappa}{k_0}\right)^2 \frac{1}{\mu_{eq}} \tag{13}$$

have to solved for ϵ_{eq} and μ_{eq}. We conclude therefrom that, for a given value of κ and subject to the thickness restrictions $|\alpha_{a,b,eq}|\, d_{a,b,eq} \lesssim 0.1$, the bilayer and a single layer are equivalent with respect to the transformation of the *x*- and *y*-components of the fields from one exterior face to the other exterior face if

$$\epsilon_{eq} = p_a\epsilon_a + p_b\epsilon_b - \left(\frac{\kappa}{k_0}\right)^2 \times \left[\frac{p_a p_b(\mu_a - \mu_b)^2}{\mu_a\mu_b(p_a\mu_a + p_b\mu_b)} + \frac{(p_a + p_b + 1)\,(p_a + p_b - 1)}{p_a\mu_a + p_b\mu_b}\right], \tag{14}$$

$$\mu_{eq} = p_a\mu_a + p_b\mu_b\,. \tag{15}$$

(ii) *p–polarization:* The only nonzero field components being H_y, E_x and E_z, the equality of the $(1,4)$ elements of $p_a\mathcal{P}_a + p_b\mathcal{P}_b$ and $\mathcal{P}_{eq}$ suffices, along with the equality of the $(4,1)$ elements of the two matrixes. For a given value of κ and subject to the thickness restrictions $|\alpha_{a,b,eq}|\, d_{a,b,eq} \lesssim 0.1$, the bilayer and a single layer are equivalent if

$$\mu_{eq} = p_a\mu_a + p_b\mu_b - \left(\frac{\kappa}{k_0}\right)^2 \times \left[\frac{p_a p_b(\epsilon_a - \epsilon_b)^2}{\epsilon_a\epsilon_b(p_a\epsilon_a + p_b\epsilon_b)} + \frac{(p_a + p_b + 1)\,(p_a + p_b - 1)}{p_a\epsilon_a + p_b\epsilon_b}\right], \tag{16}$$

$$\epsilon_{eq} = p_a\epsilon_a + p_b\epsilon_b\,. \tag{17}$$

Clearly, the constitutive parameters of the equivalent layer are functions of both p_a and p_b; and we must point out that the sum of these two ratios need not equal unity. Furthermore, except for normal incidence (i.e., $\kappa = 0$), the constitutive parameters of the equivalent layer depend on the incident linear polarization state. Finally, the constitutive parameters of the equivalent layer change with the transverse wavenumber κ.

The foregoing equations can be manipulated to yield negative values of both ϵ'_{eq} and μ'_{eq} for either

- an EN–MN bilayer $\{\epsilon'_a < 0,\, \epsilon'_b > 0,\, \mu'_a > 0,\, \mu'_b < 0\}$ or
- a MN–EN bilayer $\{\epsilon'_a > 0,\, \epsilon'_b < 0,\, \mu'_a < 0,\, \mu'_b > 0\}$.

An EN−MN bilayer is equivalent to a NPV layer for both polarization states when $\kappa = 0$, provided the condition

$$\frac{|\mu'_b|}{\mu'_a} > \frac{p_a}{p_b} > \frac{\epsilon'_b}{|\epsilon'_a|} \tag{18}$$

holds true. The inequality (18) is applicable for a MN−EN bilayer also, if the subscripts a and b are interchanged therein, i.e.,

$$\frac{|\mu'_a|}{\mu'_b} > \frac{p_b}{p_a} > \frac{\epsilon'_a}{|\epsilon'_b|}\,. \tag{19}$$

A further specialization of $p_a + p_b = 1$ leads to the inequality

$$\frac{|\mu'_b|}{\mu'_a + |\mu'_b|} > p_a > \frac{\epsilon'_b}{|\epsilon'_a| + \epsilon'_b} \tag{20}$$

for EN−MN bilayers, and

$$\frac{|\mu'_a|}{\mu'_b + |\mu'_a|} > p_b > \frac{\epsilon'_a}{|\epsilon'_b| + \epsilon'_a} \tag{21}$$

for MN−EN bilayers.

The inequalities (18)−(21) should be adequate for both s- and p-polarization states when $\kappa/k_0 \ll 1$. In general, however, a given EN−MN (or MN−EN) bilayer is equivalent to a different NPV material for a different linear polarization state and/or transverse wavenumber. Thus, the equivalence between an EN−MN (or a MN−EN) bilayer and a NPV layer is *restricted.*

The restricted equivalence has an interesting implication for perfect lenses [8]. A perfect lens of thickness $d > 0$ is defined by the fulfillment of the condition $[\boldsymbol{f}(d)] = [\boldsymbol{f}(0)]$ for all ω and κ. Because of dispersion and dissipation, at best, this condition is fulfilled approximately. Let us imagine that the condition is fulfilled by some NPV constitutive parameters for some ω and all $|\kappa| \le \hat{\kappa}$. Then, the implementation of the acceptably imperfect lens as a cascade of thin EN−MN (or MN−EN) bilayers would require that the successive bilayers have different constitutive parameters and that the entry as well as the exit faces be curved, and even those steps may not suffice. In contrast, κ is fixed for any single−mode parallel−plate waveguide, and so is the range of operating frequencies; and the emulation of a NPV material *via* EN−MN (or MN−EN) bilayers may not be onerous.

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