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GREEN'S FUNCTIONS AND BREWSTER CONDITION FOR A HALFSPACE BOUNDED BY AN ANISOTROPIC IMPEDANCE PLANE

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Dyadic Green's functions are obtained for a halfspace bounded by an anisotropic impedance plane. Using the Fresnel reflection coefficients, these functions are derived in planewave spectral forms. The Brewster condition is also obtained.

1. Introduction

In the magnetotelluric method [1-3], the electromagnetic surface impedance is measured at a number of frequencies. Some implementations of this methods employ an airborne source of electromagnetic radiation [4]; indeed, even thunderstorms are utilized as sources [5]. In his formulation, Cagniard [1] assumed the surface impedance to be a scalar, as has been pointed out also by Spichak in his recent review paper [3]. However, geomorphological studies [e.g., 6] have shown anisotropic stress fields give rise to anisotropic consolidation of the subsurface region. This implies electrical anisotropy [7]: hence, the

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anisotropic surface impedance that has been considered by Carcione *et al.* [8], and in detail by Negi and Saraf [9]. A simpler anisotropic impedance model has also been discussed by Groom and Bailey [10].

It is quite clear that the anisotropic surface impedance is definitely a function of frequency as well as the position on the terrestrial surface, and it implicitly contains geomorphological information. Analytical treatment of the ensuing complicated boundary value problem may be simply impossible. Nevertheless, insight is often gained by the analysis of simplified models, which thought provides the motivation for this communication. The objective here is to obtain a general formulation of the frequency-domain dyadic Green's function for a halfspace bounded by a plane whose impedance dyadic is independent of the position. This will be done using planewave spectral representations and the Fresnel reflection coefficients. As a bye-product of this analysis, the Brewster condition will also be obtained.

2. Dyadic Green's functions

Let the upper halfspace $z > 0$ be occupied by free space ($\mathbf{D} = \varepsilon_0\mathbf{E}$, $\mathbf{B} = \mu_0\mathbf{H}$) and bounded by the anisotropic impedance plane $z = 0$. Along with an $\exp(-i\omega t)$ time-dependence, the free space wavenumber $k_o = \omega\sqrt{(\mu_o\varepsilon_o)}$ and the intrinsic free space impedance $\eta_o = \sqrt{(\mu_o/\varepsilon_o)}$ are defined in the customary manner. Let the cartesian unit vectors be denoted by $\mathbf{u}_x$, $\mathbf{u}_y$ and $\mathbf{u}_z$.

Since the source as well as the field points lie exclusively in the upper halfspace here, the electric field $\mathbf{E}(\mathbf{r})$ can be obtained from source current densities through [11]

$$\mathbf{E}(\mathbf{r}) = i\omega\mu_o \iiint d^3\mathbf{r}'\, \underline{\underline{G}}_e(\mathbf{r},\mathbf{r}')\bullet\mathbf{J}(\mathbf{r}') - \\ - \iiint d^3\mathbf{r}'\, \underline{\underline{G}}_m(\mathbf{r},\mathbf{r}')\bullet\mathbf{K}(\mathbf{r}'), \quad z, z' \geq 0, \tag{1}$$

provided the dyadics $\underline{\underline{G}}_e(\mathbf{r},\mathbf{r}')$ and $\underline{\underline{G}}_m(\mathbf{r},\mathbf{r}')$ are known; $\mathbf{J}(\mathbf{r})$ and $\mathbf{K}(\mathbf{r})$, respectively, are the source electric and magnetic volume current densities; $\mathbf{r}$ is the field point and $\mathbf{r}'$ is the source point. Although the impedance boundary condition is usually stated in geophysical literature as the twin relations [3]

$$E_x(\mathbf{r}) = Z_{xx} H_x(\mathbf{r}) + Z_{xy} H_y(\mathbf{r}), \quad z = 0, \tag{2a}$$
$$E_y(\mathbf{r}) = Z_{yx} H_x(\mathbf{r}) + Z_{yy} H_y(\mathbf{r}), \quad z = 0, \tag{2b}$$

which can be cast also in the forms of admittance, magnetic or telluric boundary conditions, a host of developments in the electromagnetic literature can be utilized if the formulation

$$\mathbf{u}_z\times[\mathbf{u}_z\times\mathbf{E}(\mathbf{r})] = -\eta_o\, \underline{\underline{\zeta}}\bullet[\mathbf{u}_z\times\mathbf{H}(\mathbf{r})], \ z = 0, \tag{3}$$

is used [12, 13]. Here

$$\underline{\underline{\zeta}} = \zeta_{xx}\mathbf{u}_x\mathbf{u}_x + \zeta_{xy}\mathbf{u}_x\mathbf{u}_y + \zeta_{yx}\mathbf{u}_y\mathbf{u}_x + \zeta_{yy}\mathbf{u}_y\mathbf{u}_y, \tag{4}$$

is the normalised impedance dyadic; ergo, $Z_{xx} = \eta_o\zeta_{xy}$, $Z_{xy} = -\eta_o\zeta_{xx}$, $Z_{yx} = \eta_o\zeta_{yy}$ and $Z_{yy} = -\eta_o\zeta_{yx}$. We treat the impedance as independent of position $\mathbf{r}$ on the $z = 0$ plane.

From Faraday's law and (1), it follows that

$$\mathbf{H}(\mathbf{r}) = \nabla\times\iiint d^3r'\, \underline{\underline{G}}_e(\mathbf{r},\mathbf{r}')\bullet\mathbf{J}(\mathbf{r}') - \\ - \nabla\times\iiint d^3r'\, \underline{\underline{G}}_m(\mathbf{r},\mathbf{r}')\bullet\mathbf{K}(\mathbf{r}')/i\omega\mu_o, \quad z, z' \geq 0; \tag{5}$$

therefore, from (3) and (5) we must have the conditions

$$i\omega\mu_o\mathbf{u}_z\times[\mathbf{u}_z\times\underline{\underline{G}}_e(\mathbf{r},\mathbf{r}')] = -\eta_o\, \underline{\underline{\zeta}}\bullet[\mathbf{u}_z\times[\nabla\times\underline{\underline{G}}_e(\mathbf{r},\mathbf{r}')]], \ z = 0, \tag{6a}$$

and

$$i\omega\mu_o\mathbf{u}_z\times[\mathbf{u}_z\times\underline{\underline{G}}_m(\mathbf{r},\mathbf{r}')] = -\eta_o\, \underline{\underline{\zeta}}\bullet[\mathbf{u}_z\times[\nabla\times\underline{\underline{G}}_m(\mathbf{r},\mathbf{r}')]], \ z = 0, \tag{6b}$$

satisfied for the correct solution.

3. Planewave spectral decompositions

Synthesis of the Green's dyadics can be accomplished through the partitions

$$\underline{\underline{G}}_e(\mathbf{r},\mathbf{r}') = \underline{\underline{G}}_{eo}(\mathbf{r},\mathbf{r}') + \underline{\underline{G}}_{er}(\mathbf{r},\mathbf{r}'), \tag{7a}$$

and

$$\underline{\underline{G}}_m(\mathbf{r},\mathbf{r}') = \underline{\underline{G}}_{mo}(\mathbf{r},\mathbf{r}') + \underline{\underline{G}}_{mr}(\mathbf{r},\mathbf{r}'), \tag{7b}$$

where *only* $\underline{\underline{G}}_{er}$ and $\underline{\underline{G}}_{mr}$ take the impedance plane z = 0 into account. The dyadics $\underline{\underline{G}}_{eo}$ and $\underline{\underline{G}}_{mo}$ are the usual free space Green's dyadics [11, 14],

$$\underline{\underline{G}}_{eo}(\mathbf{r},\mathbf{r}') = \{\underline{\underline{I}} + \nabla\nabla/k_o^2\} \exp(ik_oR)/4\pi R \tag{8a}$$

and

$$\underline{\underline{G}}_{mo}(\mathbf{r},\mathbf{r}') = \{\nabla\times\underline{\underline{I}}\} \exp(ik_oR)/4\pi R, \tag{8b}$$

which yield the solution of Maxwell's equations if the entire space ($|z| \geq 0$) were to be vacuous; $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and $\underline{\underline{I}}$ is the identity dyadic.

The partitions (7a,b) naturally point to a similar partition of the electromagnetic field in the upper halfspace; thus,

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_o(\mathbf{r}) + \mathbf{E}_r(\mathbf{r}), \quad z \geq 0, \tag{9}$$

where

$$\mathbf{E}_o(\mathbf{r}) = i\omega\mu_o \iiint d^3r'\, \underline{\underline{G}}_{eo}(\mathbf{r},\mathbf{r}')\bullet\mathbf{J}(\mathbf{r}') - \\ -\iiint d^3r'\, \underline{\underline{G}}_{mo}(\mathbf{r},\mathbf{r}')\bullet\mathbf{K}(\mathbf{r}') \tag{10a}$$

is the primary field that is radiated by the sources into free space without taking any interfaces into consideration. In turn, this primary field is incident on the plane z = 0, thereby giving rise to the reflected field

$$\mathbf{E}_r(\mathbf{r}) = i\omega\mu_o \iiint d^3\mathbf{r}' \, \underline{\underline{G}}_{er}(\mathbf{r},\mathbf{r}') \bullet \mathbf{J}(\mathbf{r}') - \\ - \iiint d^3\mathbf{r}' \, \underline{\underline{G}}_{mr}(\mathbf{r},\mathbf{r}') \bullet \mathbf{K}(\mathbf{r}'), \quad z \geq 0. \tag{10b}$$

After defining the quantities

$$\begin{aligned}
&\mathbf{k}_{o\pm}\{\kappa_x,\kappa_y\} = \kappa_x \, \mathbf{u}_x + \kappa_y \, \mathbf{u}_y \pm k_{oz} \, \mathbf{u}_z, \\
&\mathbf{t}_{o\pm}\{\kappa_x,\kappa_y\} = -(1/\kappa)(\kappa_y \, \mathbf{u}_x - \kappa_x \, \mathbf{u}_y) = -\mathbf{k}_{o\pm} \times \mathbf{p}_{o\pm}/k_o, \\
&\mathbf{p}_{o\pm}\{\kappa_x,\kappa_y\} = -(\pm k_{oz}/\kappa \, k_o)(\kappa_x \, \mathbf{u}_x + \kappa_y \, \mathbf{u}_y) + (\kappa/k_o)\mathbf{u}_z = \\
&\qquad = \mathbf{k}_{o\pm} \times \mathbf{t}_{o\pm}/k_o, \\
&\kappa\{\kappa_x,\kappa_y\} = +\sqrt{(\kappa_x^2 + \kappa_y^2)}, \\
&k_{oz}\{\kappa_x,\kappa_y\} = +\sqrt{(k_o^2 - \kappa^2)},
\end{aligned}$$

that are functions of the horizontal wavenumbers κ_x and κ_y, the right hand sides of (8a) and (8b) can be set down as [15]

$$\underline{\underline{G}}_{eo}(\mathbf{r},\mathbf{r}') = -\mathbf{u}_z\mathbf{u}_z \, \delta(\mathbf{R}) + (i/8\pi^2) \int_{-\infty}^{\infty} d\kappa_x \int_{-\infty}^{\infty} d\kappa_y \, (1/k_{oz}) \\ [\mathbf{t}_{o\pm}\mathbf{t}_{o\pm} + \mathbf{p}_{o\pm}\mathbf{p}_{o\pm}] \exp(i \, \mathbf{k}_{o\pm} \bullet \mathbf{R}) \tag{11a}$$

and

$$\underline{\underline{G}}_{mo}(\mathbf{r},\mathbf{r}') = -(k_o/8\pi^2) \int_{-\infty}^{\infty} d\kappa_x \int_{-\infty}^{\infty} d\kappa_y \, (1/k_{oz}) \\ [\mathbf{p}_{o\pm}\mathbf{t}_{o\pm} - \mathbf{t}_{o\pm}\mathbf{p}_{o\pm}] \exp(i \, \mathbf{k}_{o\pm} \bullet \mathbf{R}), \tag{11b}$$

respectively. In (11a,b), the upper (resp. lower) sign is to be taken for $z > z'$ (resp. $z < z'$), and $z' > 0$ for this work, while $\delta(\mathbf{R})$ is the Dirac delta function.

Substitution of (11a,b) into (10a) for $z < z'$ immediately suggests that the primary field incident on the interface $z = 0$ is a continuous spectrum of plane waves. Hence, the reflected field has also to be a continuous spectrum of plane waves [16]. One can then use the Fresnel reflection coefficients $r_{tt}\{\kappa_x,\kappa_y\}$, $r_{tp}\{\kappa_x,\kappa_y\}$, $r_{pt}\{\kappa_x,\kappa_y\}$ and $r_{pp}\{\kappa_x,\kappa_y\}$ -- that must be functions of κ_x and κ_y as a result of Snel's law -- to obtain

$$\underline{\underline{G}}_{er}(\mathbf{r},\mathbf{r}') = (i/8\pi^2) \int_{-\infty}^{\infty} d\kappa_x \int_{-\infty}^{\infty} d\kappa_y \, (1/k_{oz}) \\ [r_{tt} \, \mathbf{t}_{o+}\mathbf{t}_{o-} + r_{pt} \, \mathbf{p}_{o+}\mathbf{t}_{o-} + r_{tp} \, \mathbf{t}_{o+}\mathbf{p}_{o-} + r_{pp} \, \mathbf{p}_{o+}\mathbf{p}_{o-}] \\ \exp(i \, \mathbf{k}_{o+} \bullet \mathbf{r}) \exp(-i \, \mathbf{k}_{o-} \bullet \mathbf{r}'), \quad z \geq 0, z' > 0 \tag{12a}$$

and

$$\underline{\underline{G}}_{mr}(\mathbf{r},\mathbf{r}') = -(k_o/8\pi^2)\,{}_{-\infty}\!\int^{\infty} d\kappa_x\,{}_{-\infty}\!\int^{\infty} d\kappa_y\,(1/k_{oz}) \\ [r_{tt}\,\mathbf{p}_{o+}\mathbf{t}_{o-} - r_{pt}\,\mathbf{t}_{o+}\mathbf{t}_{o-} + r_{tp}\,\mathbf{p}_{o+}\mathbf{p}_{o-} - r_{pp}\,\mathbf{t}_{o+}\mathbf{p}_{o-}] \\ \exp(i\,\mathbf{k}_{o+}\bullet\mathbf{r})\exp(-i\,\mathbf{k}_{o-}\bullet\mathbf{r}'), \quad z \geq 0, z' > 0. \quad (12b)$$

The only thing left at this stage is to ascertain the reflection coefficients r_{tt}, etc., consistent with (6a,b). It is parenthetically observed that in formalizing (12a,b), no assumptions have been made regarding the electromagnetic properties of the lower halfspace, nor of the plane $z = 0$.

4. Fresnel Reflection Coefficients

We now consider the spectral decomposition of the electromagnetic field: for a specified horizontal variation $\exp[i(\kappa_x x + \kappa_y y)]$, we must have

$$\mathbf{E}'(\mathbf{r}) = [A\,\mathbf{t}_{o-} + B\,\mathbf{p}_{o-}]\exp(i\,\mathbf{k}_{o-}\bullet\mathbf{r}) + \\ + [C\,\mathbf{t}_{o+} + D\,\mathbf{p}_{o+}]\exp(i\,\mathbf{k}_{o+}\bullet\mathbf{r}), \quad z \geq 0, \quad (13a)$$

and

$$\eta_o\mathbf{H}'(\mathbf{r}) = [A\,\mathbf{p}_{o-} - B\,\mathbf{t}_{o-}]\exp(i\,\mathbf{k}_{o-}\bullet\mathbf{r}) + \\ + [C\,\mathbf{p}_{o+} - D\,\mathbf{t}_{o+}]\exp(i\,\mathbf{k}_{o+}\bullet\mathbf{r}), \quad z \geq 0, \quad (13b)$$

where E' and H' are the spectral fields consistent with Snel's law. The coefficients A and B can be interpreted as that of a planewave incident on $z = 0$, while the coefficients C and D denote the consequently reflected planewave.

Enforcing the conditions (6a,b) is the same as enforcing

$$E'_x(\mathbf{r}) = \eta_o\,[\zeta_{xy}\,H'_x(\mathbf{r}) - \zeta_{xx}\,H'_y(\mathbf{r})], \quad z = 0, \quad (14a)$$

and

$$E'_y(\mathbf{r}) = \eta_o\,[\zeta_{yy}\,H'_x(\mathbf{r}) - \zeta_{yx}\,H'_y(\mathbf{r})], \quad z = 0; \quad (14b)$$

thus, we obtain the Fresnel reflection relationships

$$C = r_{tt} A + r_{tp} B, \qquad D = r_{pt} A + r_{pp} B, \tag{15a,b}$$

by substituting (13a,b) in (14a,b), and then solving the resulting algebraic equations simultaneously. The Fresnel coefficients work out as

$$r_{tt} = \big[-\alpha_z (1 - |\underline{\zeta}|) + \alpha_x^2 (\alpha_z^2 \zeta_{yy} - \zeta_{xx}) + \\ + \alpha_y^2 (\alpha_z^2 \zeta_{xx} - \zeta_{yy}) - \alpha_x \alpha_y (1 + \alpha_z^2)(\zeta_{xy} + \zeta_{yx})\big] / \Delta, \tag{16a}$$

$$r_{tp} = 2\alpha_z \big[\alpha_x^2 \zeta_{yx} - \alpha_y^2 \zeta_{xy} + \alpha_x \alpha_y (\zeta_{yy} - \zeta_{xx})\big] / \Delta, \tag{16b}$$

$$r_{pt} = -2\alpha_z \big[\alpha_x^2 \zeta_{xy} - \alpha_y^2 \zeta_{yx} + \alpha_x \alpha_y (\zeta_{yy} - \zeta_{xx})\big] / \Delta, \tag{16c}$$

$$r_{pp} = \big[\alpha_z (1 - |\underline{\zeta}|) + \alpha_x^2 (\alpha_z^2 \zeta_{yy} - \zeta_{xx}) + \\ + \alpha_y^2 (\alpha_z^2 \zeta_{xx} - \zeta_{yy}) - \alpha_x \alpha_y (1 + \alpha_z^2)(\zeta_{xy} + \zeta_{yx})\big] / \Delta, \tag{16d}$$

where $\alpha_x = \kappa_x/\kappa$, $\alpha_y = \kappa_y/\kappa$, $\alpha_z = k_{oz}/k_o$, $|\underline{\zeta}| = \zeta_{xx}\zeta_{yy} - \zeta_{xy}\zeta_{yx}$, and

$$\Delta = \alpha_z (1 + |\underline{\zeta}|) + \alpha_x^2 (\alpha_z^2 \zeta_{yy} + \zeta_{xx}) + \\ + \alpha_y^2 (\alpha_z^2 \zeta_{xx} + \zeta_{yy}) + \alpha_x \alpha_y (1 - \alpha_z^2)(\zeta_{xy} + \zeta_{yx}). \tag{17}$$

With these coefficients substituted into (12a,b), we have the dyadic Green's functions for a halfspace bounded by an anisotropic impedance plane.

Before continuing further, an interesting relation between the Fresnel coefficients is worth pointing out:

$$[r_{pp} - r_{tt}] / [1 - r_{tt} r_{pp} + r_{tp} r_{pt}] = (1 - |\underline{\zeta}|)/(1 + |\underline{\zeta}|), \tag{18}$$

Note should be made that the right hand side of (18) is independent of the wavenumbers, being solely dependent on the determinant of the normalised surface impedance dyadic $\underline{\zeta}$. In a manner, (18) extends a similar relationship found elsewhere for the interface of two isotropic dielectric-magnetic media [17].

5. Brewster Condition

The developments in the previous section can be utilized to give us a bonus result. This refers specifically to what is known as the Brewster angle [14, 18].

In a series of communications [19, 20, and references cited therein], the concept of the Brewster angle has been considerably enlarged into the Brewster condition, which is actually faithful to Brewster's original report [18]. Briefly, the Brewster condition arises when the reflection ratio C/D in (13a,b) becomes independent of the incidence ratio A/B.

From (15a,b) it is clear that C/D becomes decorrelated from the ratio A/B, provided κ_x and κ_y are such that

$$r_{tt}\, r_{pp} = r_{tp}\, r_{pt}, \tag{19}$$

or, equivalently,

$$0 = -\,\alpha_z\,(1 + |\underline{\zeta}|) + \alpha_x^2\,(\alpha_z^2\zeta_{yy} + \zeta_{xx}) + \\ + \alpha_y^2\,(\alpha_z^2\zeta_{xx} + \zeta_{yy}) + \alpha_x\alpha_y\,(1 - \alpha_z^2)(\zeta_{xy} + \zeta_{yx}). \tag{20}$$

Equation (20) constitutes the Brewster condition for a halfspace bounded by an anisotropic impedance plane, while the particular doublet $\{\kappa_x, \kappa_y\}$ that leads to the satisfaction of (20) should be called the Brewster wavenumber doublet. As a corollary, it follows from (18) that

$$r_{pp} - r_{tt} = (1 - |\underline{\zeta}|)/(1 + |\underline{\zeta}|) \tag{21}$$

when the Brewster condition is satisfied.

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