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Physical fractals: self-similarity and square-integrability

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The Richardson similarity dimension of a truly self-similar and square-integrable function of one variable is shown to be 3/2. This implies that many supposedly 'self-similar' functions of physical provenances are actually not so. Instead, such functions may be approximated well by truly self-similar, but non-square-integrable, functions in some restricted ranges of the independent variables.

Fractals became ubiquitous in science and technology libraries during the 1980s. The Hausdorff Besicovitch dimension of a fractal must exceed its topological dimension by definition ([1], p. 16). The Hausdorff Besicovitch dimension is difficult to obtain rigorously for many mathematical problems, not to mention for problems that derive from investigations of natural processes. However, there is the idea of a similarity dimension available for the self-similar fractals commonly studied. Mathematicians often use the similarity dimension to guess the Hausdorff Besicovitch dimension ([1], p. 37), but physicists and engineers are normally content simply to obtain the similarity dimension [2,3]. Indeed, the similarity dimension is so often invoked that reports on 'observations' of self-similarity in nature are spread all over the scientific literature of the past 10 years [3,4]. Catalysis [5], fracture toughness of metals [6], fluid dynamics [7], aerogels [8], multi-phase materials [9], geography [10] and architecture [11], are only some of the areas of scientific research in which self-similarity has been 'found'.

The rationale behind these types of studies is as follows: Let $h(x;a)$ be a self-similar function of x with a scale $a > 1$ such that

$$h(a^n x; a) = b^n h(x; a); -\infty \le x \le \infty, n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where b is some constant. Specifically, $h(ax;a) = b\, h(x;a)$, and it leads to the solution (ref. 1, p. 36)

$$h(x; a) = x^{1-D} \tag{2}$$

provided

$$b = a^{1-D} \tag{3}$$

D being the Richardson similarity dimension of $h(x;a)$. But the solution provided by Equation 2 is deceptive: It is easy to see that $h(px;a) = p^{1-D}\, h(x;a)$ is a consequence of Equation 2 for all $p \epsilon \{-\infty,\infty\}$, and can make it appear to some that a self-similar function merely expresses a power law. In contrast, Equation 1 would suggest that the relation $h(px;a) = p^{1-D} h(x;a)$ is true only if $p = a^n$.

A self-similar process is not a power-law process [12]. Nevertheless, the mode in which investigations on suspected self-similar processes are carried out is often precisely the same as that for suspected power-law ones (see e.g., [5,6,9]). Computed or measured values of the dependent variable h are plotted against the discrete values x_m of the independent variable x on a log–log graph for $x_{in} \leq x_m \leq x_{fi}$. Then, the data points are fitted, if possible, onto a straight line whose slope is equal to $(1-D)$, but the scale a remains undetermined in such studies.

Of course, the collected data are sparse because $-\infty < x_{in}$ and $x_{fi} < \infty$, and in many instances hardly more than three cycles on the abscissa of the log–log graph are used. The data windows used are so severely restricted (see, for instance, [13,14]) that the term 'self-similar' often allocated to the investigated processes appears to be unjustified and leads to the following question: What types of square-integrable functions are truly self-similar for all x? The requirement of square-integrability of a function $\eta(\zeta)$, i.e.,

$$0 < \int_{-\infty}^{\infty} d\zeta \, |\eta(\zeta)|^2 < \infty \tag{4}$$

is necessary for physical relevance, because $|\eta(\zeta)|^2$ is often taken to be proportional to energy density in the physical sciences.

Let the square-integrability condition, Equation 4, be imposed on the self-similar function $h(x;a)$ defined by Equation 1. Then,

$$E = \int_{-\infty}^{\infty} dx \, | \, h(x;a) \, |^2 = a \int_{-\infty}^{\infty} dy \, | \, h(ay;a) \, |^2 = ab^2 \int_{-\infty}^{\infty} dy \, | \, h(y;a) \, |^2 = ab^2 \, E \tag{5}$$

which can be true for a finite non-zero E only if

$$1/a = b^2 \tag{6}$$

But the relationship, expressed by Equation 3 holds for self-similar functions, and the joint solution of Equations 3 and 6 is given by

$$D = 3/2 \tag{7}$$

The upshot of this simple analysis is the answer to the question posed earlier. All square-integrable and truly self-similar functions have a Richardson similarity dimension $D = 3/2$ and satisfy the relation

$$h_E(ax;a) = a^{1-(3/2)} h_E(x;a) = a^{-1/2} h_E(x;a); \; a > 1, \; -\infty \leq x \leq \infty \tag{8}$$

Let $\mathcal{H}$ be the set of all functions satisfying Equation 1 and $\mathcal{H}_E$ be the set of all functions satisfying Equation 8, i.e.,

$$\mathcal{H} = \{h(x;a) \text{ s.t. } h(ax;a) = a^{1-D}\, h(x;a);\ a > 1,\ -\infty \le x \le \infty,\ -\infty < D < \infty\} \quad (9)$$

$$\mathcal{H}_E = \{h_E(x;a) \text{ s.t. } h_E(ax;a) = a^{-1/2}\, h_E(x;a);\ a > 1,\ -\infty \le x \le \infty\} \quad (10)$$

Then, $\mathcal{H}_E$ is a subset of $\mathcal{H}$ because
$\mathcal{H}_E = \{h(x;a) \text{ s.t. } h(ax;a) = a^{1-D}\, h(x;a)$ and

$$0 < \int_{-\infty}^{\infty} dx \mid h(x;a) \mid^2 < \infty;\ a > 1,\ -\infty \le x \le \infty,\ -\infty < D < \infty\} \quad (11)$$

These comments have a bearing on suspected self-similar functions that have physical – as distinct from mathematical – provenances. It can be easily gathered from physics literature that the estimates of the Richardson similarity dimensions of supposedly self-similar functions of one variable are rarely equal to 3/2. Moreover, while D is estimated from a narrow data window, the scale a remains usually undetermined. Finally, fractal-to-compact transitions have been observed and reported [15,16].

With the foregoing considerations in mind, I conjecture that the characteristics of the actual function $f(x)$ that is probed in any one of these studies are as follows:

(i) It possesses finite energy, i.e., $0 < \int_{-\infty}^{\infty} dx \mid f(x) \mid^2 < \infty$.

(ii) It appears to be self-similar within a restricted range: i.e., $f(x) \cong h(x;a)$ for $x_{in} \le x \le x_{fi}$, where $h(x;a) \in \mathcal{H}$, $-\infty < x_{in}$ and $x_{fi} < \infty$.

Condition (i) reiterates the physical underpinnings of $f(x)$. Condition (ii) allows $f(x)$ to appear to have a Richardson similarity dimension other than 3/2 if $f(x)$ is investigated within an appropriate range of the independent variable x. Indeed, condition (ii) allows $f(x)$ to have different values of D in different x-ranges of finite widths, in which case $f(x)$ may be dubbed *multifractal* in current parlance. Similar conclusions may be drawn for physical functions of more than one independent variables.

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