The Brewster wave-number concept for elastodynamics

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The existence of a Brewster wave number is demonstrated for the case of plane-wave reflection by a specularly smooth, planar interface joining two homogeneous, linearly elastic, isotropic solid half-spaces. Such a wave number is numerically shown to exist for two material combinations: steel–aluminum and steel–brass. The wave number can be such that plane waves are either propagating or evanescent. Since the calculated Brewster wave number is not dependent upon which of the two half-spaces contains the incident plane waves, it appears to be an intrinsic property of the bimaterial interface and hence may be useful in the characterization of such interfaces.

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INTRODUCTION

The discovery of the Brewster angle in 1812, coming at the heels of Malus' discovery of polarization in 1810, did much to establish the transverse nature of light in the next 20 years. By extensive experimentation on specularly smooth interfaces of air and isotropic dielectric substances, Brewster found that when unpolarized light is incident at a specific angle with respect to the normal to the interface, the reflected light is linearly polarized. This specific angle, later named after Brewster, is such that its tangent equals the refractive index of the dielectric material.

The method of polarization by reflection became an important tool for designers of optical instruments.[1] However, it appears that the use of semitransparent mirrors in lasers lead to a change in the definition of the Brewster angle. This new definition represented a weakening of understanding of the Brewster phenomenon, as has recently been cataloged by Lakhtakia.[2] As the original concept is pregnant with meaning, a return to the pre-World War II definition appears desirable; indeed, a number of theoretical exercises have ended up in the postulation of the idea of the Brewster wave number.[3]

The Brewster wave number can be explained as follows: Let the plane $z=0$ be the interface between two linear, homogeneous, nondiffusive electromagnetic substances. Further, let all fields be independent of the y coordinate and have an $e^{i\kappa x}$ dependence on the x coordinate.[4] From either half-space there can be two different plane eigenwaves *incident* on the interface while there will also be two plane waves *reflected* into that half-space. If κ equals the Brewster wavenumber for that interface then, (i) the ratio of amplitudes of the reflected plane eigenwaves is independent of the ratio of amplitudes of the incident plane eigenwaves, and (ii) condition (i) is satisfied regardless of incidence from the upper or lower half-spaces. It should be noted that the simpler definition of the Brewster angle is totally contained in that of the more meaningful Brewster wave number.

The mathematical unity between electromagnetics and elastodynamics[5] has prompted us to ask: Does the Brewster wave number exist for the scattering of elastic plane waves by planar solid/solid interfaces?

Plane waves of three types can exist in a homogeneous, linearly elastic, isotropic solid. Of these, the longitudinal (P) and the vertically polarized (SV) plane waves are coupled to one another at the interface $z=0$, but the horizontally polarized shear wave (SH) is decoupled from the others. Thus the Brewster phenomenon at the planar bimaterial interface of two linear isotropic solids can involve only the P and the SV plane waves. This situation is roughly parallel to that in electromagnetics, and the results of our investigation are the focus of this report.

I. ANALYSIS

Consider plane, time harmonic longitudinal (P), and transverse (SV) waves, simultaneously incident from medium I onto the plane interface $z=0$ joining two rigidly bonded, homogeneous, isotropic, and linearly elastic half-spaces, I and II (Fig. 1). Being isotropic, the half-spaces are fully characterized by their Lamé constants, λ^{I}, μ^{I}, λ^{II}, μ^{II}, and their mass densities, ρ^{I} and ρ^{II}. Consistent with the governing differential equation,

$$\mu\nabla^2\mathbf{u}+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}=\rho\frac{\partial^2\mathbf{u}}{\partial t^2}, \tag{1}$$

the incident P and SV displacement fields can be expressed as

$$\mathbf{u}_p^i=A_p\left\{\frac{\kappa\hat{\mathbf{x}}-\delta_p^{\mathrm{I}}\hat{\mathbf{z}}}{k_p^{\mathrm{I}}}\right\}e^{i\kappa x}e^{-i\delta_p^{\mathrm{I}}z},\quad z\geqslant 0, \tag{2a}$$

$$\mathbf{u}_s^i=A_s\left\{\frac{\delta_s^{\mathrm{I}}\hat{\mathbf{x}}+\kappa\hat{\mathbf{z}}}{k_s^{\mathrm{I}}}\right\}e^{i\kappa x}e^{-i\delta_s^{\mathrm{I}}z},\quad z\geqslant 0, \tag{2b}$$

where κ is the horizontal wave number which, according to Snell's law, must be the same for all of the incident, reflected, and refracted plane waves. All fields are assumed to vary with time as $e^{-i\omega t}$, which has been omitted for brevity. The vertical wave numbers are defined by $\delta_{p,s}^{\mathrm{I}} = +\{k_{p,s}^{\mathrm{I}2}-\kappa^2\}^{1/2}$, where k_p^{I} and k_s^{I} are the propagation constants of

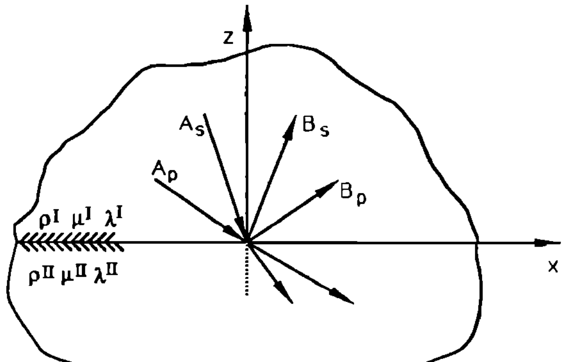

FIG. 1. Reflection of plane longitudinal (P) and transverse (SV) waves from the plane interface joining two, rigidly bonded, homogeneous, isotropic and linearly elastic half-spaces.

the longitudinal and shear waves in medium I, respectively. A real κ smaller than k_p^{I} (or k_s^{I}) ensures that the P (or SV) wave is propagating, whereas a κ larger than k_p^{I} (or k_s^{I}) corresponds to an evanescent P (or SV) wave.

Both incident plane waves will give rise to reflected P and SV waves, the displacement fields of which can be written as

$$\mathbf{u}_p^r = B_p\left\{\frac{\kappa\hat{\mathbf{x}}+\delta_p^{\mathrm{I}}\hat{\mathbf{z}}}{k_p^{\mathrm{I}}}\right\}e^{i\kappa x}e^{i\delta_p^{\mathrm{I}}z},\quad z>0, \tag{3a}$$

$$\mathbf{u}_s^r = B_s\left\{\frac{-\delta_s^{\mathrm{I}}\hat{\mathbf{x}}+\kappa\hat{\mathbf{z}}}{k_s^{\mathrm{I}}}\right\}e^{i\kappa x}e^{i\delta_s^{\mathrm{I}}z},\quad z>0. \tag{3b}$$

Transmitted fields will, of course, also be generated in the second half-space, but transmission is not of sufficient interest here to warrant further discussion. The (possibly complex) amplitudes of the reflected fields B_p, B_s can be written in terms of those of the incident fields A_p, A_s most conveniently in matrix form as

$$\begin{Bmatrix} B_p \\ B_s \end{Bmatrix} = \begin{bmatrix} R_{pp} & R_{ps} \\ R_{sp} & R_{ss} \end{bmatrix}\begin{Bmatrix} A_p \\ A_s \end{Bmatrix}, \tag{4}$$

where $R_{\alpha\beta}$ denotes the reflection factor of mode α due to incidence of mode β. Explicit expressions for these factors can be found in Ref. 7. Each of the four reflection factors $R_{\alpha\beta}$ can be obtained as the ratio of two 4×4 determinants whose elements depend on κ as well as the material properties of the two media.

If the P and SV plane waves are incident from the lower medium (II), equations analogous to Eqs. (2) and (3) are obtained where the operative propagation constants are k_p^{II} and k_s^{II}, with vertical wave numbers δ_p^{II} and δ_s^{II}, respectively. In this case, the relationship between the reflected amplitudes B_p', B_s' and the incident amplitudes A_p', A_s' can be written as

$$\begin{Bmatrix} B_p' \\ B_s' \end{Bmatrix} = \begin{bmatrix} r_{pp} & r_{ps} \\ r_{sp} & r_{ss} \end{bmatrix}\begin{Bmatrix} A_p' \\ A_s' \end{Bmatrix}. \tag{5}$$

As follows from Chen,[8] and discussed by Lakhtakia,[3] the ratio of the reflected amplitudes B_p/B_s (or B_p'/B_s') is

TABLE I. Material properties used in numerical runs.

Material	λ [GPa]	μ [GPa]	ρ [g/m^3]
Aluminum	55.3	26.0	2.7
Brass	89.2	34.0	8.5
Inconel	120.3	74.7	8.3
Plexiglas	5.4	1.3	1.1
Steel	111.8	79.9	7.8

independent of the ratio of the incident amplitudes A_p/A_s (or A_p'/A_s') if, for some wave number κ, the conditions,

$$\{R_{ss}R_{pp}-R_{sp}R_{ps}\}\big|_{\kappa=\kappa^{\mathrm{I}}}=0, \tag{6a}$$

$$\{r_{ss}r_{pp}-r_{sp}r_{ps}\}\big|_{\kappa=\kappa^{\mathrm{II}}}=0 \tag{6b}$$

are satisfied.

That this is the case can be seen if one writes the reflected amplitude ratio in the form,

$$\frac{B_p}{B_s}=\frac{R_{ps}}{R_{ss}}\left[\frac{1+\gamma\Delta}{1+\gamma'\Delta}\right], \tag{7}$$

where $\gamma=R_{pp}/R_{ps}, \gamma'=R_{sp}/R_{ss}$, and Δ has been defined as the incident amplitude ratio, A_p/A_s. It is seen that if $\gamma=\gamma'$ for some κ^{I}, then the reflected amplitude is independent of Δ and, in fact, equals $R_{ps}(\kappa^{\mathrm{I}})/R_{ss}(\kappa^{\mathrm{I}})$. Note that the condition $\gamma=\gamma'$ is precisely Eq. (6a). Similar considerations lead to Eq. 6(b).

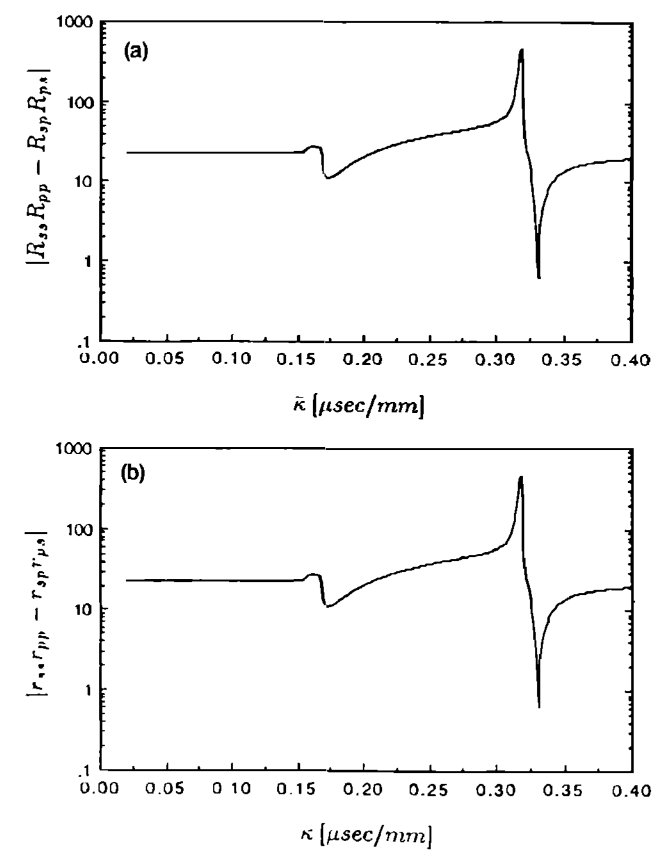

FIG. 2. Plots of (a) $|R_{ss}R_{pp}-R_{sp}R_{ps}|$, and (b) $|r_{ss}r_{pp}-r_{sp}r_{ps}|$ for the steel–aluminum combination. In (a) incidence is from the steel side, in (b) incidence is from the aluminum side.

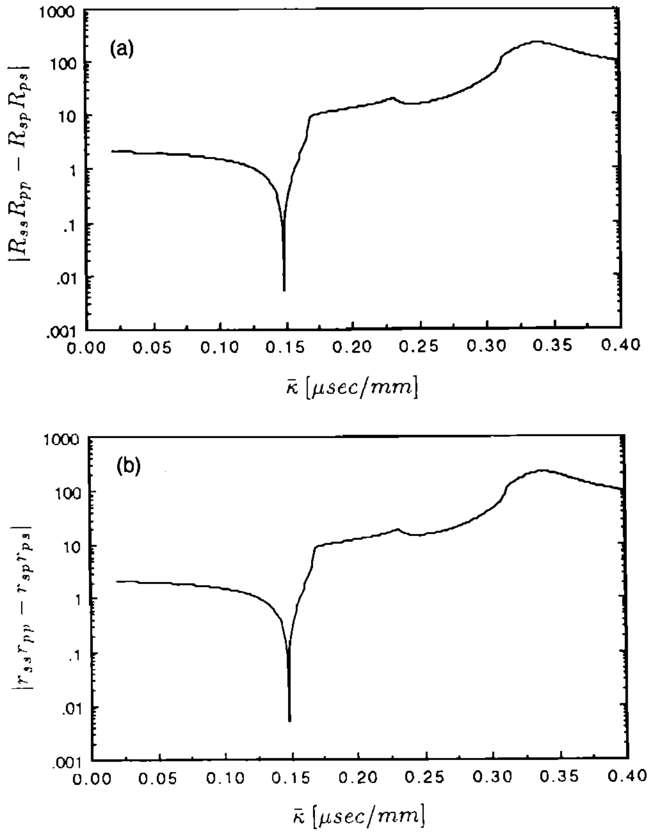

FIG. 3. Plots of (a) $|R_{ss}R_{pp}-R_{sp}R_{ps}|$, and (b) $|r_{ss}r_{pp}-r_{sp}r_{ps}|$ for the steel–brass combination. In (a) incidence is from the steel side, in (b) incidence is from the brass side.

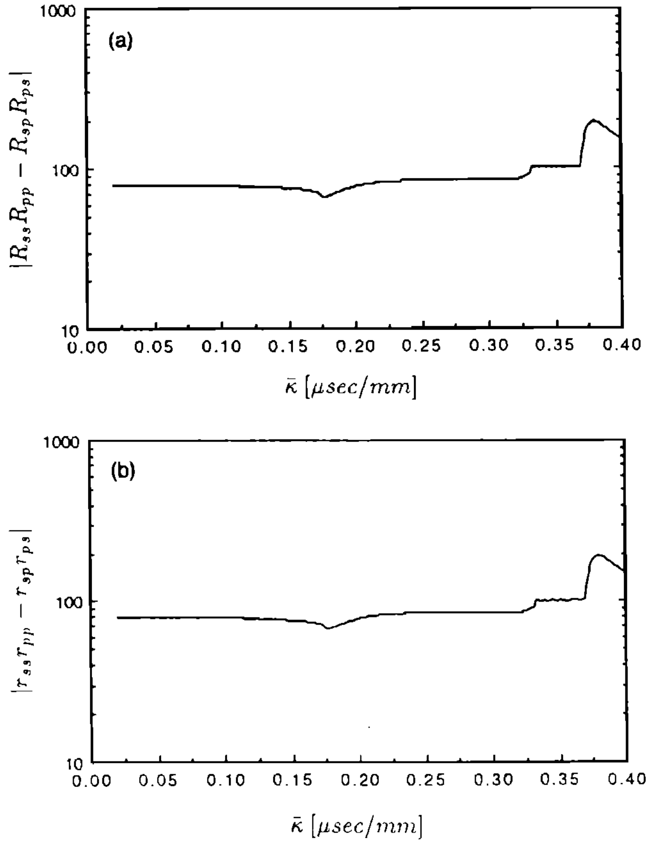

FIG. 4. Plots of (a) $|R_{ss}R_{pp}-R_{sp}R_{ps}|$, and (b) $|r_{ss}r_{pp}-r_{sp}r_{ps}|$ for the Plexiglas–Inconel combination. In (a) incidence is from the Plexiglas side, in (b) incidence is from the Inconel side.

Depending on which half-space contains the incident fields, we thus have a Brewster condition which, when satisfied, implies that the ratio of reflection amplitudes is completely *decorrelated* from the ratio of incidence amplitudes. In the numerical examples to be discussed next, it has been found, in accord with the findings of the analogous problem in electrodynamics,[3] that the conditions (6a) and (6b) are equivalent, and that, at least for the real roots found, $\kappa^{\mathrm{I}}=\kappa^{\mathrm{II}}$.

II. NUMERICAL EXAMPLES

The conditions (6a) and (6b) were evaluated for three material combinations, steel–aluminum, steel–brass, and Plexiglas–Inconel. The material properties used for each material are given in Table I. For each bimaterial configuration, incidence of P and SV waves were considered from both half-spaces. Figures 2–4 are plots of the magnitude of the quantities on the left-hand sides of Eqs. (6a) or (6b), denoted by $|R|$ and $|r|$, respectively, as a function of the horizontal slowness, $\bar{\kappa}=\kappa/\omega$. Physically, case (a) in each figure corresponds to incidence from the upper half-space, whereas case (b) corresponds to incidence from the lower half-space.

As can be seen, Eqs. (6a) and (6b) are identical conditions for each of the three cases. Furthermore, there exists, for the steel–aluminum and steel–brass combinations, a $\bar{\kappa}$ satisfying the requirements (6a) and (6b) *simultaneously*. A root finding subroutine, available from the IMSL Fortran libraries, was used to locate precisely these roots which were found to be: $\bar{\kappa}^{\mathrm{I}}=\bar{\kappa}^{\mathrm{II}}=0.332$ μs/mm and $\bar{\kappa}^{\mathrm{I}}=\bar{\kappa}^{\mathrm{II}}=0.149$ μs/mm for the steel–aluminum and steel–brass cases, respectively. No real roots were found in the range 0.001 $\mu\text{s/mm} \leqslant \bar{\kappa} \leqslant 1.2$ μs/mm for the Plexiglas–Inconel case.

It is interesting to note that the $\bar{\kappa}$ found for the steel–aluminum configuration is such that all modes are evanescent, whereas the $\bar{\kappa}$ found for the steel–brass case is such that all modes are propagating. The latter case is therefore more amenable to experimental verification. Furthermore that no values of the Brewster wave number was found (in the indicated range) for the Inconel–Plexiglas case is not surprising. In the electromagnetic case, the possibilities of none[9] or two[10] Brewster wave numbers have been reported.

III. CONCLUSIONS

The possible existence of a Brewster wave number has been demonstrated for two material combinations, and it has been shown that the Brewster wave number can be such that plane waves are either propagating or evanescent. Brewster angle spectroscopy has recently been found to be an important tool for characterizing defect levels in semiconductors.[11] In elastodynamics, it is quite possible that the concept of Brewster wave number may find application in the nondestructive characterization of interfaces since it appears to be an inherent characteristic of the bimaterial interface joining the two media, but not dependent upon which medium contains the incident fields.

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[7] B. A. Auld, *Acoustic Fields and Waves in Solids* (Kreiger, Malabar, FL, 1990), Vol. II. Note that there is a typo in Eq. (9.40c), make the change $\theta_i \rightarrow \theta$.

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