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# Trirefringent potentials for isotropic birefringent media

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Time-harmonic Lorentz vector and scalar potentials have been obtained for homogeneous biisotropic media, as also the integral equations relating source densities to the radiated potentials. It has been shown that the vector potentials are generally trirefringent, each vector potential consisting of two helical and one irrotational components. Potentials radiated by the canonical sources have been obtained, and the concept of canonical potentials has been introduced. Hertz potentials are examined, as also the Lorentz scalar potentials in a charged biisotropic region.

## 1. Introduction

Considerable research has been done in recent years on electromagnetic fields in complex media (e.g., refs. [1–3]) at microwave, millimeter wave, infrared, and optical frequencies. Among isotropic materials, the chiral (natural optically active) materials have garnered a good deal of attention [4].

But chiral media are merely special cases of the more general biisotropic media [5] that are now increasingly being investigated as well [6–8]. A number of different, but inter-convertible, sets of frequency-domain constitutive relations are available for biisotropic media [7]. In this paper we will use the Tellegen set in which [5]

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \alpha \boldsymbol{H} , \qquad \boldsymbol{B} = \beta \boldsymbol{E} + \mu \boldsymbol{H} . \tag{1}$$

Here, $\varepsilon$ and $\mu$ are the permittivity and the permeability scalars, respectively; $\alpha$ and $\beta$ are the biisotropy pseudoscalars; and an $\exp(-\mathrm{i}\omega t)$ time-dependence is implicit in this work.

Apparently, the first major investigation of time-harmonic fields in a biisotropic medium is due to Chambers [5] who showed that such a material is circularly birefringent in general. Chambers also deduced that for losslessness, $\varepsilon$ and $\mu$ should be purely real, while $\alpha$ and $\beta$ must be complex conjugates of one another. From Krowne [9], it can be deduced that the Tellegen medium is reciprocal if $\alpha = -\beta$, which is the special case of natural optical activity. Monzon [6] has obtained the infinite-medium dyadic Green's functions, Huygen's principle, and surface integral equations, while Lakhtakia [10] has obtained volume integral equations.

In free space it is known that the Lorentz potentials are, to quote van Bladel [11], "less discontinuous" than the fields, and have application in radiation formulations. Time-harmonic Lorentz potentials for radiation in a natural optically active medium have been investigated by Gvozdev and Serdyukov [12], and Varadan et al. [13], while time-harmonic Lorentz potentials in a sourceless Tellegen medium were given by Chambers [5].

However, the author is unaware of a more comprehensive study for potentials in homogeneous

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biisotropic media, and this provides the motivation for what follows. Integral equations relating source current densities to the radiated Lorentz vector potentials are derived and the potentials are shown to be generally trirefringent, consisting of two helical and one irrotational components. The concept of canonical potentials in biisotropic media is introduced. In addition, Hertz potentials are examined, as also the Lorentz scalar potentials in a charged biisotropic region.

## 2. Preliminaries

We begin with the source-incorporated time-harmonic Maxwell's equations [14]

$$\nabla \cdot \boldsymbol{D} = \rho_e , \qquad \nabla \cdot \boldsymbol{B} = \rho_m , \qquad \nabla \times \boldsymbol{E} - i\omega \boldsymbol{B} = -\boldsymbol{K} , \qquad \nabla \times \boldsymbol{H} + i\omega \boldsymbol{D} = \boldsymbol{J} , \tag{2}$$

in which $\{\rho_e, \boldsymbol{J}\}$ are the source *electric* {charge, current} densities, while $\{\rho_m, \boldsymbol{K}\}$ are the source *magnetic* {charge, current} densities. Using eq. (1) now for all space, we obtain the dyadic differential equations

$$[\nabla \times \nabla \times \boldsymbol{I} - (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} - \gamma_1\gamma_2 \boldsymbol{I}] \cdot \boldsymbol{E}(\boldsymbol{r}) = i\omega[\mu \boldsymbol{J}(\boldsymbol{r}) - \alpha \boldsymbol{K}(\boldsymbol{r})] - \nabla \times \boldsymbol{K}(\boldsymbol{r}) , \tag{3a}$$

$$[\nabla \times \nabla \times \boldsymbol{I} - (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} - \gamma_1\gamma_2 \boldsymbol{I}] \cdot \boldsymbol{H}(\boldsymbol{r}) = i\omega[\varepsilon \boldsymbol{K}(\boldsymbol{r}) - \beta \boldsymbol{J}(\boldsymbol{r})] + \nabla \times \boldsymbol{J}(\boldsymbol{r}) , \tag{3b}$$

in which $\boldsymbol{I}$ is the identity dyadic. Here and hereafter,

$$\gamma_1 = (\omega/2)[i(\beta - \alpha) + \sqrt{4(\varepsilon\mu - \alpha\beta) - (\beta - \alpha)^2}] , \tag{4a}$$

and

$$\gamma_2 = (\omega/2)[i(\alpha - \beta) + \sqrt{4(\varepsilon\mu - \alpha\beta) - (\beta - \alpha)^2}] \tag{4b}$$

are wavenumbers, whose real parts are always positive. For later purposes, we will define a third wavenumber

$$\gamma_3 = \sqrt{\gamma_1\gamma_2} = \omega\sqrt{\varepsilon\mu - \alpha\beta} , \tag{4c}$$

which also must have its real part positive. Furthermore, we will assume that $4\varepsilon\mu \neq (\beta + \alpha)^2$ in what follows.

Since the infinite-medium Green's dyadics for the Tellegen media are known [6], the solutions of the differential equations can be obtained everywhere as [10]:

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_h(\boldsymbol{r}) + \iiint d^3r' [\boldsymbol{G}_{ee}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') + \boldsymbol{G}_{em}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{K}(\boldsymbol{r}')] , \tag{5a}$$

and

$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_h(\boldsymbol{r}) + \iiint d^3r' [\boldsymbol{G}_{me}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') + \boldsymbol{G}_{mm}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{K}(\boldsymbol{r}')] . \tag{5b}$$

Here, $\boldsymbol{E}_h(\boldsymbol{r})$ and $\boldsymbol{H}_h(\boldsymbol{r})$ are the respective (complementary) solutions of the homogeneous counterparts (i.e., right side $= \boldsymbol{0}$) of eqs. (3a) and (3b). The four Green's functions utilised for the particular solutions of eqs. (3a) and (3b) are

$$\boldsymbol{G}_{ee}(\boldsymbol{r}, \boldsymbol{r}') = i\omega\mu[\boldsymbol{G}_1(\boldsymbol{R}) + \boldsymbol{G}_2(\boldsymbol{R})] , \tag{6a}$$

$$\boldsymbol{G}_{em}(\boldsymbol{r}, \boldsymbol{r}') = -(\gamma_1 + i\omega\alpha)\boldsymbol{G}_1(\boldsymbol{R}) + (\gamma_2 - i\omega\alpha)\boldsymbol{G}_2(\boldsymbol{R}) , \tag{6b}$$

$$\boldsymbol{G}_{me}(\boldsymbol{r}, \boldsymbol{r}') = (\gamma_1 - i\omega\beta)\boldsymbol{G}_1(\boldsymbol{R}) - (\gamma_2 + i\omega\beta)\boldsymbol{G}_2(\boldsymbol{R}) , \tag{6c}$$

$$\boldsymbol{G}_{mm}(\boldsymbol{r}, \boldsymbol{r}') = i\omega\varepsilon[\boldsymbol{G}_1(\boldsymbol{R}) + \boldsymbol{G}_2(\boldsymbol{R})] , \tag{6d}$$

with

$$\boldsymbol{G}_1(\boldsymbol{R}) = [\gamma_1 \boldsymbol{I} + \nabla\nabla/\gamma_1 + \nabla \times \boldsymbol{I}] g_1(R)/(\gamma_1 + \gamma_2) , \tag{7a}$$

$$\boldsymbol{G}_2(\boldsymbol{R}) = [\gamma_2 \boldsymbol{I} + \nabla\nabla/\gamma_2 - \nabla \times \boldsymbol{I}] g_2(R)/(\gamma_1 + \gamma_2) , \tag{7b}$$

$$\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$$

and

$$g_n(R) = \exp(\mathrm{i}\gamma_n R)/4\pi R , \qquad n = 1, 2, 3 . \tag{7c}$$

As per current usage for chiral media [15], it will be proper to call the sum

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = \boldsymbol{G}_1(\boldsymbol{R}) + \boldsymbol{G}_2(\boldsymbol{R}) , \tag{8a}$$

the Green's dyadic for electromagnetic fields in the Tellegen biisotropic medium as it satisfies the differential equation

$$[\nabla \times \nabla \times \boldsymbol{I} - (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} - \gamma_1\gamma_2 \boldsymbol{I}] \cdot \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') = \boldsymbol{I}\, \delta(\boldsymbol{R}) , \tag{8b}$$

$\delta(\boldsymbol{R})$ being the Dirac delta.

## 3. Lorentz vector potentials

Following refs. [5] and [16], and after using eq. (1) in eq. (2), by direct substitution it can be verified that the decomposition

$$\boldsymbol{E} = \mathrm{i}\omega(\boldsymbol{A} + \nabla\nabla\cdot\boldsymbol{A}/\gamma_3^2) - \varepsilon^{-1}[\nabla \times \boldsymbol{F} + \mathrm{i}\omega\alpha(\boldsymbol{F} + \nabla\nabla\cdot\boldsymbol{F}/\gamma_3^2)] , \tag{9a}$$

$$\boldsymbol{H} = \mu^{-1}[\nabla \times \boldsymbol{A} - \mathrm{i}\omega\beta(\boldsymbol{A} + \nabla\nabla\cdot\boldsymbol{A}/\gamma_3^2)] + \mathrm{i}\omega(\boldsymbol{F} + \nabla\nabla\cdot\boldsymbol{F}/\gamma_3^2) , \tag{9b}$$

$$\boldsymbol{D} = \mu^{-1}[\alpha\nabla \times \boldsymbol{A} + \mathrm{i}(\gamma_3^2\boldsymbol{A} + \nabla\nabla\cdot\boldsymbol{A})/\omega] - \nabla \times \boldsymbol{F} , \tag{9c}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} - \varepsilon^{-1}[\beta\nabla \times \boldsymbol{F} - \mathrm{i}(\gamma_3^2\boldsymbol{F} + \nabla\nabla\cdot\boldsymbol{F})/\omega] , \tag{9d}$$

can be made, provided the Lorentz vector *magnetic* potential $\boldsymbol{A}$ satisfies the differential equation

$$[\nabla^2\boldsymbol{I} + (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} + \gamma_3^2\boldsymbol{I}] \cdot \boldsymbol{A}(\boldsymbol{r}) = -\mu\boldsymbol{J}(\boldsymbol{r}) , \tag{10a}$$

and the Lorentz vector *electric* potential $\boldsymbol{F}$ is the solution of the equation

$$[\nabla^2\boldsymbol{I} + (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} + \gamma_3^2\boldsymbol{I}] \cdot \boldsymbol{F}(\boldsymbol{r}) = -\varepsilon\boldsymbol{K}(\boldsymbol{r}) . \tag{10b}$$

The similarities of the left hand sides of eqs. (10a) and (10b) should be noted.

Solving eqs. (10a) or (10b) requires a Green's function $\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}')$ that satisfies the equation

$$[\nabla^2\boldsymbol{I} + (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} + \gamma_3^2\boldsymbol{I}] \cdot \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}') = -\boldsymbol{I}\, \delta(\boldsymbol{R}) . \tag{11}$$

Then,

$$\boldsymbol{A}(\boldsymbol{r}) = \boldsymbol{A}_{\mathrm{h}}(\boldsymbol{r}) + \mu \int\int\int \mathrm{d}^3r'\, \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') , \tag{12a}$$

and

$$\boldsymbol{F}(\boldsymbol{r}) = \boldsymbol{F}_{\mathrm{h}}(\boldsymbol{r}) + \varepsilon \int\int\int \mathrm{d}^3r'\, \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}') \cdot \boldsymbol{K}(\boldsymbol{r}') , \tag{12b}$$

with $\boldsymbol{A}_{\mathrm{h}}(\boldsymbol{r})$ and $\boldsymbol{F}_{\mathrm{h}}(\boldsymbol{r})$, respectively, being the solutions of the homogeneous counterparts of (10a) and (10b).

The inversion of eq. (11) has been done using the Fourier technique described in refs. [1, 12, 13].

There being no particular need to give the detailed derivation here, we content ourselves with stating the final result:

$$\boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}') = \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}') - \gamma_3^{-2}\nabla\nabla g_3(R) = \boldsymbol{G}_1(\boldsymbol{R}) + \boldsymbol{G}_2(\boldsymbol{R}) - \gamma_3^{-2}\nabla\nabla g_3(R) . \tag{13}$$

## 4. Discussion

### 4.1. Hertz potentials

Hertz potentials are often used in place of Lorentz potentials in free space [17]. For the time-harmonic treatment of a biisotropic medium, we can define a Hertz *electric* potential

$$\boldsymbol{\Pi}_e(\boldsymbol{r}) = i\omega\gamma_3^{-2}\boldsymbol{A}(\boldsymbol{r}) , \tag{14a}$$

and a Hertz *magnetic* potential

$$\boldsymbol{\Pi}_m(\boldsymbol{r}) = i\omega\gamma_3^{-2}\boldsymbol{F}(\boldsymbol{r}) . \tag{14b}$$

These Hertz potentials satisfy, respectively, the relations

$$[\nabla^2\boldsymbol{I} + (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} + \gamma_3^2\boldsymbol{I}] \cdot \boldsymbol{\Pi}_e(\boldsymbol{r}) = -i\omega\mu\gamma_3^{-2}\boldsymbol{J}(\boldsymbol{r}) , \tag{15a}$$

and

$$[\nabla^2\boldsymbol{I} + (\gamma_1 - \gamma_2)\nabla \times \boldsymbol{I} + \gamma_3^2\boldsymbol{I}] \cdot \boldsymbol{\Pi}_m(\boldsymbol{r}) = -i\omega\varepsilon\gamma_3^{-2}\boldsymbol{K}(\boldsymbol{r}) . \tag{15b}$$

### 4.2. Lorentz scalar potentials

The decomposition (9a–9d) has been given somewhat disingenuously since it involves only vector potentials, but it is the usual practice (e.g., refs. [5, 11]) to prescribe in addition a Lorentz scalar *electric* potential $v(\boldsymbol{r}) \propto \nabla\cdot \boldsymbol{A}(\boldsymbol{r})$ and a Lorentz scalar *magnetic* potential $\varpi(\boldsymbol{r}) \propto \nabla\cdot \boldsymbol{F}(\boldsymbol{r})$ . In the present instance, we have [5, 13]

$$v(\boldsymbol{r}) = -i\omega\gamma_3^{-2}\nabla\cdot \boldsymbol{A}(\boldsymbol{r}) = -\nabla\cdot \boldsymbol{\Pi}_e(\boldsymbol{r}) , \tag{16a}$$

and

$$\varpi(\boldsymbol{r}) = -i\omega\gamma_3^{-2}\nabla\cdot \boldsymbol{F}(\boldsymbol{r}) = -\nabla\cdot \boldsymbol{\Pi}_m(\boldsymbol{r}) . \tag{16b}$$

These scalar potentials can be shown to satisfy, respectively, the scalar differential equations

$$[\nabla^2 + \gamma_3^2]v(\boldsymbol{r}) = -(\omega^2\gamma_3^{-2}\mu)\rho_e(\boldsymbol{r}) , \tag{17a}$$

and

$$[\nabla^2 + \gamma_3^2]\varpi(\boldsymbol{r}) = -(\omega^2\gamma_3^{-2}\varepsilon)\rho_m(\boldsymbol{r}) . \tag{17b}$$

The scalar Green's functions for the scalar Helmholtz equation are well known (e.g., refs. [1, 11]); ergo,

$$v(\boldsymbol{r}) = v_h(\boldsymbol{r}) + (\omega^2\gamma_3^{-2}\mu)\int\int\int d^3r'\; g_3(R)\rho_e(\boldsymbol{r}') , \tag{18a}$$

$$\varpi(\boldsymbol{r}) = \varpi_{\mathrm{h}}(\boldsymbol{r}) + (\omega^2\gamma_3^{-2}\varepsilon)\int\int\int \mathrm{d}^3r'\ g_3(R)\rho_{\mathrm{m}}(\boldsymbol{r}')\ , \tag{18b}$$

with the subscripts '$h$' denoting the complementary solutions of eqs. (17a,b).

### 4.3. Charged biisotropic region

The field of a charged particle in a material medium is often examined (e.g., ref. [18]), for which purpose we look at the electro-magnetic-static case: $\omega = 0$ and $\boldsymbol{J} = \boldsymbol{K} = \boldsymbol{0}$ in eq. (2). Then, the Maxwell curl equations yield $\boldsymbol{E} = -\nabla v$ and $\boldsymbol{H} = -\nabla\varpi$. Substitution of these representations into the Maxwell divergence equations for the Tellegen medium gives

$$(\varepsilon\mu - \alpha\beta)\nabla^2 v(\boldsymbol{r}) = \alpha\rho_{\mathrm{m}}(\boldsymbol{r}) - \mu\rho_{\mathrm{e}}(\boldsymbol{r})\ , \tag{19a}$$

and

$$(\varepsilon\mu - \alpha\beta)\nabla^2 \varpi(\boldsymbol{r}) = \beta\rho_{\mathrm{e}}(\boldsymbol{r}) - \varepsilon\rho_{\mathrm{m}}(\boldsymbol{r})\ . \tag{19b}$$

Quite clearly, finding $v(\boldsymbol{r})$ and $\varpi(\boldsymbol{r})$ constitutes a well-posed problem if and only if $\varepsilon\mu \neq \alpha\beta$. The solutions of (19a,b) require the static Green's function [19]

$$g_{\mathrm{P}}(\boldsymbol{r}, \boldsymbol{r}') = 1/4\pi|\boldsymbol{r} - \boldsymbol{r}'|\ , \tag{20}$$

for Poisson's equation, and we have

$$v(\boldsymbol{r}) = v_{\mathrm{h}}(\boldsymbol{r}) - (\varepsilon\mu - \alpha\beta)^{-1}\int\int\int \mathrm{d}^3r'\ g_{\mathrm{P}}(\boldsymbol{r}, \boldsymbol{r}')[\alpha\rho_{\mathrm{m}}(\boldsymbol{r}') - \mu\rho_{\mathrm{e}}(\boldsymbol{r}')]\ , \tag{21a}$$

$$\varpi(\boldsymbol{r}) = \varpi_{\mathrm{h}}(\boldsymbol{r}) - (\varepsilon\mu - \alpha\beta)^{-1}\int\int\int \mathrm{d}^3r'\ g_{\mathrm{P}}(\boldsymbol{r}, \boldsymbol{r}')[\beta\rho_{\mathrm{e}}(\boldsymbol{r}') - \varepsilon\rho_{\mathrm{m}}(\boldsymbol{r}')]\ . \tag{21b}$$

Here, the subscript '$h$' denotes the complementary solutions of eqs. (19a) and (19b). Kong [18] had looked at the case of $\alpha = \beta$ with $\rho_{\mathrm{m}}(\boldsymbol{r}) = 0$, and we find complete agreement with his results for that special case.

We will have occasion to use scalar potentials only sparingly in the remainder of this paper, so the term *potential(s)* will imply *vector potentials(s)* hereonwards unless otherwise noted.

### 4.4. Refringences

The most obvious distinction between the fields and the potentials is in their respective refringences. Using a diagonalisation transform [8, 10] for $\boldsymbol{E}_{\mathrm{h}}$ and $\boldsymbol{H}_{\mathrm{h}}$ in eqs. (5a) and (5b), and separating the contributions of $\boldsymbol{G}_1(\boldsymbol{R})$ and $\boldsymbol{G}_2(\boldsymbol{R})$, it is clear that these can be rewritten as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{Q}_1(\boldsymbol{r}) + \boldsymbol{Q}_2(\boldsymbol{r})\ , \tag{22a}$$

and

$$\boldsymbol{H}(\boldsymbol{r}) = \mathrm{i}[\eta_1^{-1}\boldsymbol{Q}_1(\boldsymbol{r}) + \eta_2^{-1}\boldsymbol{Q}_2(\boldsymbol{r})]\ . \tag{22b}$$

In this breakup, the impedances are given as

$$\eta_1 = -\omega\mu/(\gamma_1 - \mathrm{i}\omega\beta)\ , \tag{23a}$$

$$\eta_2 = \omega\mu/(\gamma_2 + \mathrm{i}\omega\beta)\ , \tag{23b}$$

while the Beltrami fields are [20–22]

$$Q_1(r) = Q_{1h}(r) + \iiint d^3r'\, G_1(R) \cdot [i\omega\mu J(r') - (\gamma_1 + i\omega\alpha)K(r')] , \tag{24a}$$

$$Q_2(r) = Q_{2h}(r) + \iiint d^3r'\, G_2(R) \cdot [i\omega\mu J(r') + (\gamma_2 - i\omega\alpha)K(r')] . \tag{24b}$$

We note the relation $\eta_1 \eta_2 = -\mu/\varepsilon$; furthermore, for the transformation (22a,b) to be possible [23] it is essential that $\eta_1 \neq \eta_2$, which condition is the same as $4\varepsilon\mu \neq (\beta + \alpha)^2$.

As shown above, the fields $E$ and $H$ can be broken up into two components, each with its own distinct wavenumber (i.e., $\gamma_1$ and $\gamma_2$) in general. On the other hand, a simple look at the right hand side of eq. (13) suffices to show that the Lorentz vector potentials $A$ and $F$ are trirefringent in general, there being three distinct wavenumbers (i.e., $\gamma_1$, $\gamma_2$ and $\gamma_3$) in $A(r, r')$. Indeed, $A$ (and $F$ similarly) can be represented by the sum

$$A(r) = A_1(r) + A_2(r) + A_3(r) , \tag{25a}$$

where

$$A_1(r) = A_{1h}(r) + \mu \iiint d^3r'\, G_1(R) \cdot J(r') , \tag{25b}$$

$$A_2(r) = A_{2h}(r) + \mu \iiint d^3r'\, G_2(R) \cdot J(r') , \tag{25c}$$

$$A_3(r) = A_{3h}(r) - \mu\gamma_3^{-2} \iiint d^3r' [\nabla\nabla g_3(R)] \cdot J(r') . \tag{25d}$$

Furthermore, the scalar potentials $\upsilon$ and $\varpi$ have only the wavenumber $\gamma_3$.

The deductions made in the previous paragraph apply to general biisotropic media. The special case of $\alpha = \beta$, however, stands out; then, it is easy to see that $\gamma_1 = \gamma_2 = \gamma_3 = \omega\sqrt{\varepsilon\mu - \alpha^2}$. Thus, the fields as well as the potentials are unirefringent for $\alpha = \beta$, but the medium is nonreciprocal unless $\alpha = 0$ as well.

### 4.5. Helicities

Returning to the general case, from the properties

$$\nabla \times G_1(R) = \gamma_1 G_1(R) , \qquad \nabla \times G_2(R) = -\gamma_2 G_2(R)$$

for $R \neq 0$, it is easily shown that

$$\nabla \times Q_1(r) = \gamma_1 Q_1(r) , \tag{26a}$$

$$\nabla \times Q_2(r) = -\gamma_2 Q_2(r) \tag{26b}$$

in a sourceless region. This proves that the two electromagnetic field components are handed with opposite helicities. But

$$\nabla \times A_1(r) = \gamma_1 A_1(r) , \qquad \nabla \times A_2(r) = -\gamma_2 A_2(r) , \qquad \nabla \times A_3(r) = 0 \tag{27}$$

in a sourceless region. Thus, although two of the three components of each vector potential are also handed with opposite helicities, the third component is purely irrotational. The situation of the potentials in a biisotropic medium, as shown by eq. (27), bears a marked resemblance to that of the fields in a plasma [24].

### 4.6. Singularities

The singularities of the various Green's functions provide a further comparison of the fields and the potentials. The contribution of the term $\nabla\nabla g_n(R)$ in the limit $R \to 0$ has been extensively discussed by Yaghjian [25], and recently also by Weiglhofer [26], in the distributional sense. Thus [25],

$$\lim_{R\to 0} \nabla\nabla g_n(R) = -\boldsymbol{L}\,\delta(\boldsymbol{R})\,, \qquad n = 1, 2, 3\,, \tag{28a}$$

$$\lim_{R\to 0} \boldsymbol{G}_n(\boldsymbol{R}) = -\boldsymbol{L}\,\delta(\boldsymbol{R})/\gamma_n(\gamma_1 + \gamma_2)\,, \qquad n = 1, 2\,, \tag{28b}$$

in which $\boldsymbol{L}$ is a real symmetric dyadic with unit trace. This dyadic $\boldsymbol{L}$ depends on the shape of the principal volume used to enclose the singular point. We now obtain that, in the distributional sense,

$$\lim_{r\to r'} \boldsymbol{G}_{\mathrm{ee}}(\boldsymbol{r}, \boldsymbol{r}') = -\mathrm{i}\omega\mu\boldsymbol{L}\,\delta(\boldsymbol{R})/\gamma_1\gamma_2\,, \tag{29a}$$

$$\lim_{r\to r'} \boldsymbol{G}_{\mathrm{em}}(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}\omega\alpha\boldsymbol{L}\,\delta(\boldsymbol{R})/\gamma_1\gamma_2\,, \tag{29b}$$

$$\lim_{r\to r'} \boldsymbol{G}_{\mathrm{me}}(\boldsymbol{r}, \boldsymbol{r}') = \mathrm{i}\omega\beta\boldsymbol{L}\,\delta(\boldsymbol{R})/\gamma_1\gamma_2\,, \tag{29c}$$

$$\lim_{r\to r'} \boldsymbol{G}_{\mathrm{mm}}(\boldsymbol{r}, \boldsymbol{r}') = -\mathrm{i}\omega\varepsilon\boldsymbol{L}\,\delta(\boldsymbol{R})/\gamma_1\gamma_2\,, \tag{29d}$$

but

$$\lim_{r\to r'} \boldsymbol{A}(\boldsymbol{r}, \boldsymbol{r}') = \boldsymbol{0}\,\delta(\boldsymbol{R})\,, \tag{30a}$$

$$\lim_{R\to 0} g_3(R) = 0\,\delta(\boldsymbol{R})\,, \tag{30b}$$

with $\boldsymbol{0}$ being the null dyadic. This shows that the potentials are "less discontinuous" than the fields even in biisotropic media.

### 4.7. Canonical sources

Just as the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ were decomposed into the Beltrami fields $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$, we can set up source current densities $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ such that [22]

$$\boldsymbol{K}(\boldsymbol{r}) = -\boldsymbol{S}_1(\boldsymbol{r}) - \boldsymbol{S}_2(\boldsymbol{r})\,, \tag{31a}$$

$$\boldsymbol{J}(\boldsymbol{r}) = \mathrm{i}[\eta_1^{-1}\boldsymbol{S}_1(\boldsymbol{r}) + \eta_2^{-1}\boldsymbol{S}_2(\boldsymbol{r})]\,. \tag{31b}$$

Then,

$$\nabla \times \boldsymbol{Q}_1(\boldsymbol{r}) = \gamma_1\boldsymbol{Q}_1(\boldsymbol{r}) + \boldsymbol{S}_1(\boldsymbol{r})\,, \tag{32a}$$

$$\nabla \times \boldsymbol{Q}_2(\boldsymbol{r}) = -\gamma_2\boldsymbol{Q}_2(\boldsymbol{r}) + \boldsymbol{S}_2(\boldsymbol{r})\,, \tag{32b}$$

showing that $\boldsymbol{S}_1$ (respectively $\boldsymbol{S}_2$) is the *canonical* source of electromagnetic fields with wavenumber $\gamma_1$ (respectively $\gamma_2$).

Even though the fields $\boldsymbol{E}$ and $\boldsymbol{H}$ due to the canonical source $\boldsymbol{S}_1$ (or $\boldsymbol{S}_2$) alone are unirefringent, the corresponding potentials $\boldsymbol{A}$ and $\boldsymbol{F}$ are still trirefringent. To see that, consider the case when $\boldsymbol{S}_2(\boldsymbol{r}) \equiv 0$ everywhere and eqs. (31a,b) are used: then eqs. (19a,b) yield

$$\boldsymbol{Q}_1(\boldsymbol{r}) - \boldsymbol{Q}_{1\mathrm{h}}(\boldsymbol{r}) = -\omega\eta_1^{-1}(\mu + \varepsilon\eta_1^2)\iiint \mathrm{d}^3r'\,\boldsymbol{G}_1(\boldsymbol{R})\cdot\boldsymbol{S}_1(\boldsymbol{r}')\,, \tag{33a}$$

$$\boldsymbol{Q}_2(\boldsymbol{r}) - \boldsymbol{Q}_{2\mathrm{h}}(\boldsymbol{r}) = \boldsymbol{0}\,, \tag{33b}$$

and eqs. (12a,b) give

$$A(r) - A_h(r) = i\mu\eta_1^{-1} \int\int d^3r' \, \mathbf{A}(r, r') \cdot S_1(r') , \tag{33c}$$

along with the relationship

$$F(r) - F_h(r) = -i\eta_2^{-1}[A(r) - A_h(r)] . \tag{33d}$$

There is nothing in eqs. (33c,d) to indicate that the potentials due only to the canonical source $S_1$ are unirefringent.

Likewise, when $S_1(r) \equiv 0$ everywhere, we have

$$Q_1(r) - Q_{1h}(r) = 0 , \tag{34a}$$

$$Q_2(r) - Q_{2h}(r) = -\omega\eta_2^{-1}(\mu + \epsilon\eta_2^2) \int\int\int d^3r' \, G_2(R) \cdot S_2(r') , \tag{34b}$$

$$A(r) - A_h(r) = i\mu\eta_2^{-1} \int\int d^3r' \, \mathbf{A}(r, r') \cdot S_2(r') , \tag{34c}$$

$$F(r) - F_h(r) = -i\eta_1^{-1}[A(r) - A_h(r)] \tag{34d}$$

due to $S_2$ alone. Again, the electromagnetic fields due solely to the canonical source $S_2$ are unirefringent, but the potentials are not.

### 4.8. Canonical potentials

In $S_1$ and $S_2$ we have canonical current densities*, and in $Q_1$ and $Q_2$ we have canonical fields. The developments of section 4.7 can be taken a step further: the potentials $A$ and $F$ can be replaced by canonical potentials.

Using eqs. (10a,b) and (31a,b) we can define canonical potentials $P_1$ and $P_2$ such that

$$F(r) = \epsilon[P_1(r) + P_2(r)] , \tag{35a}$$

$$A(r) = -i\mu[\eta_1^{-1}P_1(r) + \eta_2^{-1}P_2(r)] . \tag{35b}$$

These canonical potentials satisfy the differential equation

$$[\nabla^2 I + (\gamma_1 - \gamma_2)\nabla \times I + \gamma_3^2 I] \cdot P_n(r) = S_n(r) , \qquad n = 1, 2 . \tag{36}$$

By substituting eqs.(35a,b) in eqs. (9a,b), and after making use of eqs. (22a,b), we can obtain the representations

$$Q_1 = -\nabla \times P_1 - \gamma_2 P_1 - \nabla\nabla \cdot P_1/\gamma_1 , \tag{37a}$$

$$Q_2 = -\nabla \times P_2 + \gamma_1 P_2 + \nabla\nabla \cdot P_2/\gamma_2 , \tag{37b}$$

that can be easily verified by using eqs. (32a,b) and (36). We stress here that though the canonical field $Q_n$ (for $n = 1, 2$) generated by the canonical source $S_n$ is unirefringent, the corresponding canonical potential $P_n$ is still trirefringent.

*By following procedures similar to those in ref. [22] for chiral media, we can have canonical charge densities as well.

## 5. Summary

We have obtained the time-harmonic Lorentz vector potentials of the electric and the magnetic type for homogeneous biisotropic media, and have derived integral equations relating source current densities to the radiated potentials. Even though the fields are generally birefringent, it has been shown that the potentials are generally trirefringent. It has also been shown that each vector potential consists of two helical and one irrotational components. We have also obtained the potentials radiated by the canonical sources, and introduced the concept of canonical potentials in biisotropic media. In addition, Hertz potentials are examined, as also the Lorentz scalar potentials in a charged biisotropic region.

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