International Journal of Modern Physics C, Vol. 3, No. 3 (1992) 583–603

STRONG AND WEAK FORMS OF THE METHOD OF MOMENTS AND THE COUPLED DIPOLE METHOD FOR SCATTERING OF TIME-HARMONIC ELECTROMAGNETIC FIELDS

AKHLESH LAKHTAKIA

Department of Engineering Science and Mechanics, The Pennsylvania State University, University Park, PA 16802-1401, USA

Received 31 March 1992

Algorithms based on the method of moments (MOM) and the coupled dipole method (CDM) are commonly used to solve electromagnetic scattering problems. In this paper, the strong and the weak forms of both numerical techniques are derived for bianisotropic scatterers. The two techniques are shown to be fully equivalent to each other, thereby defusing claims of superiority often made for the charms of one technique over the other. In the final section, reductions of the algorithms for isotropic dielectric scatterers are explicitly given.

1. Introduction

The passage of an electromagnetic signal through a non-vacuous region is accompanied by the causally-related disturbance of the centers of charges and currents from their equilibrium positions. In the language of time-harmonic microscopic electromagnetism, it is therefore convenient to think that an electromagnetic wave induces electric and magnetic multipoles that oscillate in definite phase relationship with the incident wave and re-radiate energy in all directions. In time-harmonic macroscopic terms, the electric and the magnetic multipoles give rise to the polarization and the magnetization fields in matter,[1,2] and the re-radiated energy is called scattered radiation.

The frequency-domain interaction of electromagnetic waves with matter is governed by the time-harmonic form of Maxwell postulates[3] which appear to embrace a wide range of electromagnetic phenomena.[4] Now, the importance of boundary value problems in the understanding of physical processes cannot be exaggerated: not only have these problems been applied for explaining such an everyday occurrence as the blue color of the sky, but have been used as scientific tools for scanning the human anatomy, for determining the composition of materials, for the remote sensing of the terrestrial surface, and so on. Yet, exact solutions of boundary value problems are known only for very simple shapes.[5,6]

With the advent of various new techniques that rely heavily on digital computers, however, problems involving more complicated geometries have been treated. Apart from some low-[7] or high-[8] frequency methods and the T-matrix method,[9] numerical techniques involve the discretization of the region occupied by the scatterer.[10] This is generally true whether a differential formulation is used or an integrodifferential formalism.

The method of moments (MOM),[11–13] as applied to an inhomogeneous dielectric scatterer is an approach based on the evaluation of a volume integral equation over the region occupied by the scatterer. This region is partitioned into a number of subregions (generally cubical), the electromagnetic field in each subregion being represented by local basis functions. In this manner, the volume integral equation is converted into a set of simultaneous algebraic equations that are solved using standard procedures.

Whereas the MOM is an *actual field* formalism, the coupled dipole method (CDM) is based on the concept of an *exciting field*. The CDM was formulated very heuristically by Purcell and Pennypacker[14] for dielectric scatterers. Although it used to be thought of as a microscopic approach, it was shown by Lakhtakia[15] that the CDM has, at most, a semi-microscopic basis. Indeed, the operational basis for applying the CDM to boundary value problems is totally macroscopic. Both the MOM and the CDM were recently extended to bianisotropic scatterers; furthermore, what are called later as their respective *weak* forms, were shown to be equivalent.[16]

In the immediate past, there has been interest shown[17,18] in comparing numerical results obtained from implementing CDM and MOM algorithms on digital computers. In some of these studies, the *weak* form of the CDM is compared with a *strong* form of the MOM. If a comparison is not done with comparable levels of sophistication, any conclusions regarding the superiority of one technique over another can only be regarded with bemused contempt. Moreover, efforts have been reported[19,20] to improve the Purcell–Pennypacker algorithm of the CDM.

These developments catalyzed the present paper, its first aim being to provide a uniform and self-consistent derivation of the MOM and the CDM for very general boundary value problems. The second aim is to identify the *strong* and the *weak* forms of, as well as to demonstrate the formal equivalence of, both numerical techniques. The plan of this paper is as follows: In Sec. 2, the geometry of the boundary value problem is described, followed in the next section by a brief foray into the characteristics of homogeneous bianisotropic media. Two integrals, of seminal importance for the identification of the strong and the weak forms, are discussed in Sec. 4. Coupled volume integral equations are set up in Sec. 5, the MOM algebraic equations are derived in Sec. 6, and the CDM equations in Sec. 7. In the next section, remarks are made on the strong and the weak forms of the methods; in Sec. 9, formulae for scattering and absorption quantities are given. Finally, in Sec. 10 the reduction of both techniques for isotropic dielectric scatterers is given.

2. The Scattering Geometry

As is schematically illustrated in Fig. 1, let all space be divided into two mutually-disjoint regions, $V_{\rm int}$ and $V_{\rm ext}$, that are distinguishable from each other by the occupancy of matter. The region $V_{\rm ext}$ is vacuous; hence,

$$\mathbf{D}(\mathbf{x}) = \varepsilon_0 \mathbf{E}(\mathbf{x}),\ \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{H}(\mathbf{x}),\ \mathbf{x} \in V_{\rm ext}\,. \qquad \text{(1a,b)}$$

The region $V_{\rm int}$ is filled with a general, linear, possibly inhomogeneous, non-diffusive bianisotropic continuum with frequency-dependent $[\exp(-i\omega t)]$ constitutive equations[21]

$$\mathbf{D}(\mathbf{x}) = \varepsilon_0 \left[\underline{\boldsymbol{\varepsilon}}_r(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \underline{\boldsymbol{\xi}}_r(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x})\right],\ \mathbf{x} \in V_{\rm int}\,, \qquad \text{(2a)}$$

$$\mathbf{B}(\mathbf{x}) = \mu_0 \left[\underline{\boldsymbol{\zeta}}_r(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \underline{\boldsymbol{\mu}}_r(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x})\right],\ \mathbf{x} \in V_{\rm int}\,, \qquad \text{(2b)}$$

where $\underline{\boldsymbol{\varepsilon}}_r(\mathbf{x})$ is the relative permittivity dyadic, $\underline{\boldsymbol{\mu}}_r(\mathbf{x})$ is the relative permeability dyadic, while $\underline{\boldsymbol{\xi}}_r(\mathbf{x})$ and $\underline{\boldsymbol{\zeta}}_r(\mathbf{x})$ represent the magnetoelectric dyadics.

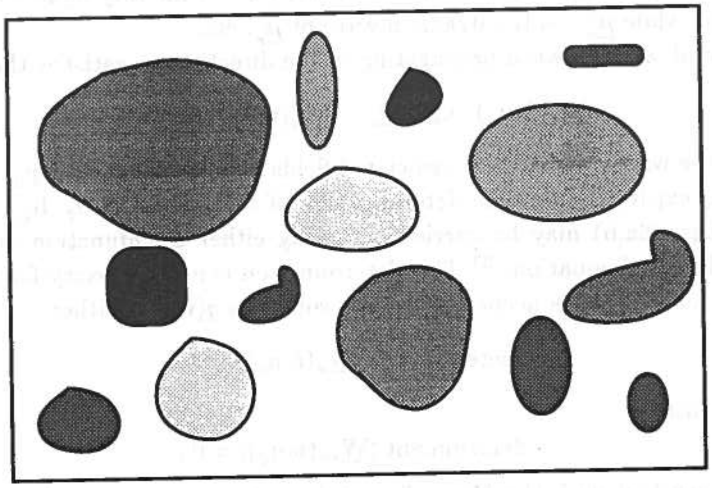

Fig. 1. Schematic of the scattering problem. The unshaded region $V_{\rm ext}$ extends to infinity in all directions, while the shaded regions collectively constitute $V_{\rm int}$.

There is no requirement that $V_{\rm int}$ be a simply-connected convex region; however, the boundary surface that separates $V_{\rm int}$ from $V_{\rm ext}$ must be at least once-differentiable everywhere to enable the unambiguous prescription of a unit normal at every point on this surface. Furthermore, the maximum linear extent of $V_{\rm int}$ must be bounded so that only the region $V_{\rm ext}$ extends out to infinity in all directions.

3. Homogeneous Bianisotropic Media

Before continuing, it pays to attend to the homogeneous counterparts of the inhomogeneous equations (2a,b). Such a medium, described by the Tellegen relations

$$\mathbf{D}(\mathbf{x}) = \varepsilon_0 \left[\underline{\boldsymbol{\varepsilon}}_r \cdot \mathbf{E}(\mathbf{x}) + \underline{\boldsymbol{\xi}}_r \cdot \mathbf{H}(\mathbf{x})\right], \quad \mathbf{B}(\mathbf{x}) = \mu_0 \left[\underline{\boldsymbol{\zeta}}_r \cdot \mathbf{E}(\mathbf{x}) + \underline{\boldsymbol{\mu}}_r \cdot \mathbf{H}(\mathbf{x})\right], \tag{3a,b}$$

in the frequency domain, occurs readily in nature. Physically possible forms, consistent with Lorentz covariance and causality, of the constitutive dyadics in Eqs. (3a,b) were discussed at length by Post.[1]

Using the Maxwell curl postulates in the frequency domain, it can be shown that the electric and the magnetic fields in a source-free homogeneous bianisotropic medium satisfy the dyadic differential equations

$$\underline{\mathbf{W}}_e(\nabla) \cdot \mathbf{E}(\mathbf{x}) = \underline{\mathbf{0}}, \quad \underline{\mathbf{W}}_m(\nabla) \cdot \mathbf{H}(\mathbf{x}) = \underline{\mathbf{0}}, \tag{4a,b}$$

where the dyadic differential operators are given by

$$\underline{\mathbf{W}}_e(\nabla) = \left[\nabla \times \underline{\mathbf{I}} + i\omega\varepsilon_0 \underline{\boldsymbol{\xi}}_r\right] \cdot \underline{\boldsymbol{\mu}}_r^{-1} \cdot \left[\nabla \times \underline{\mathbf{I}} - i\omega\mu_0 \underline{\boldsymbol{\zeta}}_r\right] - k_0^2 \underline{\boldsymbol{\varepsilon}}_r, \tag{5a}$$

$$\underline{\mathbf{W}}_m(\nabla) = \left[\nabla \times \underline{\mathbf{I}} - i\omega\mu_0 \underline{\boldsymbol{\zeta}}_r\right] \cdot \underline{\boldsymbol{\varepsilon}}_r^{-1} \cdot \left[\nabla \times \underline{\mathbf{I}} + i\omega\varepsilon_0 \underline{\boldsymbol{\xi}}_r\right] - k_0^2 \underline{\boldsymbol{\mu}}_r, \tag{5b}$$

$k_0 = \omega\sqrt{(\mu_0\varepsilon_0)}$ is the free space wave number, $\underline{\mathbf{I}}$ is the identity dyadic and $\underline{\mathbf{0}}$ is the null dyadic, while $\underline{\boldsymbol{\mu}}_r^{-1}$ is the dyadic inverse of $\underline{\boldsymbol{\mu}}_r$, etc.

In general, a plane wave propagating in the direction $\mathbf{u}_q$ satisfies the equations

$$\underline{\mathbf{W}}_e(iq\mathbf{u}_q) \cdot \mathbf{e}_q = \underline{\mathbf{0}}, \quad \underline{\mathbf{W}}_m(iq\mathbf{u}_q) \cdot \mathbf{h}_q = \underline{\mathbf{0}}, \tag{6a,b}$$

with q as the wave number, the associated fields being $\mathbf{E}(\mathbf{x}) = \mathbf{e}_q \exp[iq\mathbf{u}_q \cdot \mathbf{x}]$ and $\mathbf{H}(\mathbf{x}) = \mathbf{h}_q \exp[iq\mathbf{u}_q \cdot \mathbf{x}]$. The determination of the triads $\{q, \mathbf{e}_q, \mathbf{h}_q\}$ for a given $\mathbf{u}_q$ from Eqs. (6a,b) may be carried out using either eigenfunction analysis[22] or matrix differential equations.[23] This determination is not necessary for the present purposes, and all that is needed is the two solutions $q(\mathbf{u}_q)$ of either

$$\text{determinant}\,[\underline{\mathbf{W}}_e(iq\mathbf{u}_q)] = 0 \tag{7a}$$

or, equivalently,

$$\text{determinant}\,[\underline{\mathbf{W}}_m(iq\mathbf{u}_q)] = 0, \tag{7b}$$

that are consistent with the Maxwell postulates for a given $\mathbf{u}_q$; it is possible that the two solutions are identical. By $q^\dagger$ is denoted the maximum of $|q(\mathbf{u}_q)|$ over all directions $\mathbf{u}_q$.

4. Dyadic Green's Function and Self-Integrals

It is also advisable to consider beforehand a couple of integrals that involve the free space dyadic Green's function[6,22]

$$\underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = [\underline{\mathbf{I}} + (1/k_0^2)\nabla\nabla]\, g(\mathbf{x}, \mathbf{x}') \tag{8a}$$

and its circulation

$$\underline{\underline{\mathbf{H}}}(\mathbf{x}, \mathbf{x}') = \nabla \times \underline{\underline{\mathbf{G}}}(\mathbf{x}, \mathbf{x}') = [\nabla g(\mathbf{x}, \mathbf{x}')] \times \underline{\underline{\mathbf{I}}}\,, \tag{8b}$$

where

$$g(\mathbf{x}, \mathbf{x}') = (1/4\pi)\{\exp[ik_0|\mathbf{x} - \mathbf{x}'|]/|\mathbf{x} - \mathbf{x}'|\} \tag{8c}$$

is the scalar Green's function for the three-dimensional scalar Helmholtz equation. It is noted that $\underline{\underline{\mathbf{G}}}(\mathbf{x}, \mathbf{x}')$ is of the order $1/|\mathbf{x} - \mathbf{x}'|^3$ for $|\mathbf{x} - \mathbf{x}'| \cong 0$, and becomes singular at $\mathbf{x} = \mathbf{x}'$, this singularity requiring special attention. The dyadic $\underline{\underline{\mathbf{H}}}(\mathbf{x}, \mathbf{x}')$ is also singular at $\mathbf{x} = \mathbf{x}'$, but as it is of only the order $1/|\mathbf{x} - \mathbf{x}'|^2$ for $|\mathbf{x} - \mathbf{x}'| \cong 0$, its singularity is integrable per a lemma of Kellogg.[24]

The first of these two integrals is given as

$$\mathbf{a}_1(\mathbf{x}_0) = \iiint_V d^3\mathbf{x}'\,\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}')\,, \tag{9a}$$

where V is the region bounded by the surface S, as shown in Fig. 2. This integral is not problematic if $\mathbf{x}_0$ were to lie outside V. However, what will be needed in the hereafter is that $\mathbf{x}_0$ be a distinguished point lying inside V. As shown by Wang,[25] following up on Fikioris,[26] Eq. (9a) is transformed to

$$\begin{aligned}\mathbf{a}_1(\mathbf{x}_0) = &\iiint_{V-V_0} d^3\mathbf{x}'\,\{\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}')\}\\ &+ \iiint_{V_0} d^3\mathbf{x}'\,\{\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}') - \underline{\underline{\mathbf{G}}}_s(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}_0)\}\\ &- (1/k_0^2)\iint_{S_0} d^2\mathbf{x}'\,\mathbf{u}_n'\,\{(\mathbf{x}' - \mathbf{x}_0)/4\pi|\mathbf{x}' - \mathbf{x}_0|^3\} \cdot \mathbf{b}(\mathbf{x}_0)\,,\end{aligned} \tag{9b}$$

where V_0 is an exclusionary region bounded by the surface S_0, as shown in Fig. 2; $\mathbf{u}_n'$ is the unit normal to S_0, pointing away from V_0, at the point $\mathbf{x}' \in S_0$; while

$$\underline{\underline{\mathbf{G}}}_s(\mathbf{x}, \mathbf{x}') = (1/k_0^2)\nabla\nabla\{1/4\pi|\mathbf{x} - \mathbf{x}'|\}\,. \tag{10}$$

As Fikioris[26] noted, the exclusionary region V_0 should be small but not infinitesimal, and it must be wholly contained within V. Moreover, there is no requirement that S_0 be a miniature copy of S.

Now, we substitute $\mathbf{b}(\mathbf{x}')$ by $\{[\mathbf{b}(\mathbf{x}') - \mathbf{b}(\mathbf{x}_0)] + \mathbf{b}(\mathbf{x}_0)\}$ in the second integral on the right-hand side of Eq. (10) to obtain

$$\begin{aligned}\mathbf{a}_1(\mathbf{x}_0) = &\iiint_{V-V_0} d^3\mathbf{x}'\,\{\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}')\} + \iiint_{V_0} d^3\mathbf{x}'\,\{\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') \cdot [\mathbf{b}(\mathbf{x}') - \mathbf{b}(\mathbf{x}_0)]\}\\ &+ \iiint_{V_0} d^3\mathbf{x}'\,\{[\underline{\underline{\mathbf{G}}}(\mathbf{x}_0, \mathbf{x}') - \underline{\underline{\mathbf{G}}}_s(\mathbf{x}_0, \mathbf{x}')] \cdot \mathbf{b}(\mathbf{x}_0)\}\\ &- (1/k_0^2)\iint_{S_0} d^2\mathbf{x}'\,\mathbf{u}_n'\,\{(\mathbf{x}' - \mathbf{x}_0)/4\pi|\mathbf{x}' - \mathbf{x}_0|^3\} \cdot \mathbf{b}(\mathbf{x}_0)\,.\end{aligned} \tag{9c}$$

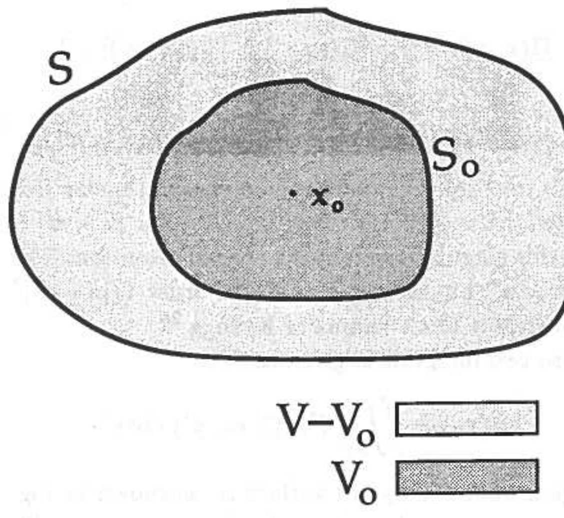

Fig. 2. For the evaluation of the integral in Eq. (9a) when $\mathbf{x}' \in V$ and $\mathbf{x}_0 \in V$.

When the integrals of the type (9a) are needed in the following sections, $\mathbf{b}(\mathbf{x}) \equiv \mathbf{b}(\mathbf{x}_0)\ \forall \mathbf{x} \in V$ is assumed because V is supposed to have electrically small dimensions. On using this long-wavelength approximation, (9c) reduces to

$$\begin{aligned}\mathbf{a}_1(\mathbf{x}_0) \cong & \iiint_{V-V_0} d^3\mathbf{x}' \left\{\underline{\mathbf{G}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}_0)\right\} \\ & + \iiint_{V_0} d^3\mathbf{x}' \left\{[\underline{\mathbf{G}}(\mathbf{x}_0, \mathbf{x}') - \underline{\mathbf{G}}_s(\mathbf{x}_0, \mathbf{x}')] \cdot \mathbf{b}(\mathbf{x}_0)\right\} \\ & - (1/k_0^2) \iint_{S_0} d^2\mathbf{x}'\, \mathbf{u}_n' \left\{(\mathbf{x}' - \mathbf{x}_0)/4\pi|\mathbf{x}' - \mathbf{x}_0|^3\right\} \cdot \mathbf{b}(\mathbf{x}_0) . \end{aligned} \quad (9d)$$

Since V is now electrically small, but V_0 need not be infinitesimal, we comfortably set $V = V_0$ and $S = S_0$ to have

$$\mathbf{a}_1(\mathbf{x}_0) \cong [\underline{\mathbf{M}} - (1/k_0^2)\underline{\mathbf{L}}] \cdot \mathbf{b}(\mathbf{x}_0) , \quad (11a)$$

where

$$\underline{\mathbf{M}} = \iiint_V d^3\mathbf{x}' \left\{[G(\mathbf{x}_0, \mathbf{x}') - \underline{\mathbf{G}}_s(\mathbf{x}_0, \mathbf{x}')]\right\} , \quad (11b)$$

$$\underline{\mathbf{L}} = \iint_S d^2\mathbf{x}'\, \mathbf{u}_n' \left\{(\mathbf{x}' - \mathbf{x}_0)/4\pi|\mathbf{x}' - \mathbf{x}_0|^3\right\} . \quad (11c)$$

The evaluation of $\underline{\mathbf{M}}$ can be accomplished numerically for a variety of V shapes using the coordinate-free expansions

$$\underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = [(\underline{\mathbf{I}} - \mathbf{u}_X\mathbf{u}_X) + (i/k_0|\mathbf{X}|) \\ (1 + i/k_0|\mathbf{X}|)(\underline{\mathbf{I}} - 3\mathbf{u}_X\mathbf{u}_X)]\, g(\mathbf{x}, \mathbf{x}'), \ |\mathbf{X}| \neq 0\,, \tag{12a}$$

$$\underline{\mathbf{G}}_s(\mathbf{x}, \mathbf{x}') = (i/k_0|\mathbf{X}|)^2(\underline{\mathbf{I}} - 3\mathbf{u}_X\mathbf{u}_X)]/4\pi|\mathbf{X}|, \ |\mathbf{X}| \neq 0\,, \tag{12b}$$

where $\mathbf{X} = \mathbf{x} - \mathbf{x}'$ and $\mathbf{u}_X = \mathbf{X}/|\mathbf{X}|$. The evaluation of the depolarization dyadic $\underline{\mathbf{L}}$ is equally easy on a computer. However, analytical expressions for $\underline{\mathbf{L}}$ are given by Lakhtakia[27] for the quite general ellipsoidal shapes; see also Yaghjian.[28]

The second integral

$$\mathbf{a}_2(\mathbf{x}_0) = \iiint_V d^3\mathbf{x}' \,\{\underline{\mathbf{H}}(\mathbf{x}_0, \mathbf{x}') \cdot \mathbf{b}(\mathbf{x}')\}\,, \quad \mathbf{x}_0 \in V\,, \tag{13}$$

has an integrable singularity as mentioned above. Upon using the long-wavelength approximation utilized for $\mathbf{a}_1(\mathbf{x}_0)$, $\mathbf{a}_2(\mathbf{x}_0)$ may be estimated as

$$\mathbf{a}_2(\mathbf{x}_0) \cong \underline{\mathbf{N}} \cdot \mathbf{b}(\mathbf{x}_0)\,, \tag{14a}$$

where the dyadic

$$\underline{\mathbf{N}} = \iiint_V d^3\mathbf{x}' \,\underline{\mathbf{H}}(\mathbf{x}_0, \mathbf{x}')\,. \tag{14b}$$

In the computation of the integral (14b), the coordinate-free expression

$$\underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}') = (ik_0 - 1/|\mathbf{X}|) g(\mathbf{x}, \mathbf{x}') \mathbf{u}_X \times \underline{\mathbf{I}} \tag{15}$$

may be used.

If V is extremely small in electrical size, the simplifications

$$\mathbf{a}_1(\mathbf{x}_0) \cong -(1/k_0^2)\underline{\mathbf{L}} \cdot \mathbf{b}(\mathbf{x}_0), \quad \mathbf{a}_2(\mathbf{x}_0) \cong \underline{\mathbf{0}} \cdot \mathbf{b}(\mathbf{x}_0)\,, \tag{16a,b}$$

are permissible and lead to the weak forms of the MOM and the CDM.

5. Coupled Volume Integral Equations

After these mathematical preliminaries we return to the scattering problem at hand. The Maxwell curl postulates, in the absence of any externally impressed sources, are given in V_{ext} as

$$\nabla \times \mathbf{E}(\mathbf{x}) - i\omega\mu_0\mathbf{H}(\mathbf{x}) = \mathbf{0}, \ \nabla \times \mathbf{H}(\mathbf{x}) + i\omega\varepsilon_0\mathbf{E}(\mathbf{x}) = \mathbf{0}, \ \mathbf{x} \in V_{\text{ext}}\,; \tag{17a,b}$$

and, similarly, in V_{int} as

$$\nabla \times \mathbf{E}(\mathbf{x}) - i\omega\mu_0 \left[\underline{\zeta}_r(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \underline{\mu}_r(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x})\right] = \mathbf{0}\,, \quad \mathbf{x} \in V_{\text{int}}\,, \tag{18a}$$

$$\nabla \times \mathbf{H}(\mathbf{x}) + i\omega\varepsilon_0[\underline{\varepsilon}_r(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \underline{\xi}_r(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x})] = \mathbf{0}\,, \quad \mathbf{x} \in V_{\text{int}}\,, \tag{18b}$$

with $\mathbf{0}$ being the null vector.

As per the equivalence principle, we replace matter in V_{int} by electric and magnetic volume current densities, $\mathbf{J}(\mathbf{x})$ and $\mathbf{K}(\mathbf{x})$, that radiate into V_{ext}. Therefore, the Maxwell curl postulates everywhere are rewritten as[29]

$$\nabla \times \mathbf{E}(\mathbf{x}) - i\omega\mu_0 \mathbf{H}(\mathbf{x}) = -\mathbf{K}(\mathbf{x}), \quad \nabla \times \mathbf{H}(\mathbf{x}) + i\omega\varepsilon_0 \mathbf{E}(\mathbf{x}) = \mathbf{J}(\mathbf{x}), \quad \mathbf{x} \in V_{\text{int}} + V_{\text{ext}}\,, \tag{19a,b}$$

where

$$\mathbf{J}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{K}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in V_{\text{ext}}\,, \tag{20a,b}$$

$$\mathbf{J}(\mathbf{x}) = i\omega\varepsilon_0[\{\mathbf{I} - \underline{\varepsilon}_r(\mathbf{x})\} \cdot \mathbf{E}(\mathbf{x}) - \xi_r(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x})], \quad \mathbf{x} \in V_{\text{int}}\,, \tag{21a}$$

$$\mathbf{K}(\mathbf{x}) = i\omega\mu_0[-\underline{\zeta}_r(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) + \{\underline{\mathbf{I}} - \underline{\mu}_r(\mathbf{x})\} \cdot \mathbf{H}(\mathbf{x})], \quad \mathbf{x} \in V_{\text{int}}\,. \tag{21b}$$

The solutions of the differential equations (19a,b) are given as the coupled volume integral equations[29]

$$\mathbf{E}(\mathbf{x}) - \mathbf{E}_{\text{inc}}(\mathbf{x}) = \iiint_{V_{\text{int}}} d^3\mathbf{x}' \{ i\omega\mu_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}') - \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}(\mathbf{x}')\}, \quad \mathbf{x} \in V_{\text{int}} + V_{\text{ext}}\,, \tag{22a}$$

and

$$\mathbf{H}(\mathbf{x}) - \mathbf{H}_{\text{inc}}(\mathbf{x}) = \iiint_{V_{\text{int}}} d^3\mathbf{x}' \{ i\omega\varepsilon_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') + \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}') \cdot \mathbf{J}(\mathbf{x}')\}, \quad \mathbf{x} \in V_{\text{int}} + V_{\text{ext}}\,. \tag{22b}$$

The fields $\mathbf{E}_{\text{inc}}(\mathbf{x})$ and $\mathbf{H}_{\text{inc}}(\mathbf{x})$ are the solutions of the sourceless (i.e. right side $\equiv \mathbf{0}$) counterparts of the differential equations (19a,b), and represent the electromagnetic field that exists in $V_{\text{int}} + V_{\text{ext}}$ if $\underline{\varepsilon}_r(\mathbf{x}) = \underline{\mathbf{I}}$, $\underline{\mu}_r(\mathbf{x}) = \underline{\mathbf{I}}$, $\underline{\zeta}_r(\mathbf{x}) = \underline{\mathbf{0}}$ and $\underline{\xi}_r(\mathbf{x}) = \underline{\mathbf{0}}$. The integral equations (22a) and (22b) are utilized in setting up the MOM and the CDM, and this commonality reaffirms the algorithmic equivalence of the two.

6. The Method of Moments

Although the method of moments, used widely in the electromagnetics community, has grown increasingly sophisticated in the last decade,[30,31] a simple enough version provides for the conversion of the integral equations (22a,b) into algebraic equations with facile ease. The scatterer region V_{int} is considered as the union of non-overlapping simply-connected subregions V_m, $(m = 1, 2, \ldots, M)$, each bounded by a closed surface S_m, so that $V_{\text{int}} = \sum_m V_m$. The requirements are as follows: (i) each boundary surface S_m, that separates V_m from other parts of V_{int} and/or V_{ext}, is at least once-differentiable; (ii) the subregion V_m is homogeneous so that

$$\underline{\varepsilon}_r(\mathbf{x}) = \underline{\varepsilon}_{r,m}\,, \ \underline{\mu}_r(\mathbf{x}) = \underline{\mu}_{r,m}\,, \ \underline{\zeta}_r(\mathbf{x}) = \underline{\zeta}_{r,m}\,, \ \underline{\xi}_r(\mathbf{x}) = \underline{\xi}_{r,m}\,, \ \mathbf{x} \in V_m\,; \tag{23}$$

and (iii) the maximum linear extent $2a_m$ of V_m is such that $k_o a_m < 0.1$ and $q_m^{\dagger} a_m < 0.1$. In these stipulations, the MOM and the CDM are also identical.

The second and the third requirements permit a long-wavelength approximation.[29,32] Let the volumetric capacity of V_m be denoted by $\nu_m = \iiint_{V_m} d^3\mathbf{x}$. Let $\mathbf{x}_m$ denote a distinguished point lying inside V_m, and let $\mathbf{E}_m = \mathbf{E}(\mathbf{x}_m)$ and $\mathbf{H}_m = \mathbf{H}(\mathbf{x}_m)$ be the *actual* fields at that point; then, we use the approximations $\mathbf{E}(\mathbf{x}) \cong \mathbf{E}_m$ and $\mathbf{H}(\mathbf{x}) \cong \mathbf{H}_m$ for all $\mathbf{x} \in \mathbf{V}_m$. Now by setting $\mathbf{x} = \mathbf{x}_k$ in Eqs. (22a,b) the $2M$ vector equations are obtained,

$$\begin{aligned}\mathbf{E}_k - i\omega\mu_0[\underline{\mathbf{M}}_k - (1/k_0^2)\underline{\mathbf{L}}_k]\cdot\mathbf{J}(\mathbf{x}_k) + \underline{\mathbf{N}}_k\cdot\mathbf{K}(\mathbf{x}_k) &= \mathbf{E}_{\mathrm{inc}}(\mathbf{x}_k)\\ &+ \sum_{m,m\neq k}\nu_m\{i\omega\mu_0\underline{\mathbf{G}}(\mathbf{x}_k,\,\mathbf{x}_m)\cdot\mathbf{J}(\mathbf{x}_m)\\ &- \underline{\mathbf{H}}(\mathbf{x}_k,\,\mathbf{x}_m)\cdot\mathbf{K}(\mathbf{x}_m)\},\ k=1,\,2,\,\dots,\,M\,, \end{aligned}\tag{24a}$$

$$\begin{aligned}\mathbf{H}_k - i\omega\varepsilon_0[\underline{\mathbf{M}}_k - (1/k_0^2)\underline{\mathbf{L}}_k]\cdot\mathbf{K}(\mathbf{x}_k) - \underline{\mathbf{N}}_k\cdot\mathbf{J}(\mathbf{x}_k) &= \mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k)\\ &+ \sum_{m,m\neq k}\nu_m\{i\omega\varepsilon_0\underline{\mathbf{G}}(\mathbf{x}_k,\,\mathbf{x}_m)\cdot\mathbf{K}(\mathbf{x}_m)\\ &+ \underline{\mathbf{H}}(\mathbf{x}_k,\,\mathbf{x}_m)\cdot\mathbf{J}(\mathbf{x}_m)\},\ k=1,\,2,\,\dots,\,M\,. \end{aligned}\tag{24b}$$

In obtaining Eqs. (24a,b), each integration on the subregion V_m, $m \neq k$, was done very simply by evaluating the specific integrand at $\mathbf{x}_m$ and multiplying it by the volumetric capacity ν_m. The integrals $\iiint_{V_k} d^3\mathbf{x}'\underline{\mathbf{G}}(\mathbf{x}_k,\,\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}')$ and $\iiint_{V_k} d^3\mathbf{x}'\underline{\mathbf{G}}(\mathbf{x}_k,\,\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}')$ were estimated using the formula (11a), while $\iiint_{V_k} d^3\mathbf{x}'\underline{\mathbf{H}}(\mathbf{x}_k,\,\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}')$ and $\iiint_{V_k} d^3x'\underline{\mathbf{H}}(\mathbf{x}_k,\,\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}')$ were estimated using the formula (13).

Using Eqs. (21a,b) and (23), Eqs. (24a,b) are transformed into 6M simultaneous algebraic equations containing 6M complex scalar unknowns that are constituted by the cartesian components of $\mathbf{E}_m$ and $\mathbf{H}_m$. These equations are compactly stated as

$$\mathbf{E}_{\mathrm{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\dots,M\}}[\underline{\mathbf{A}}_{km}\cdot\mathbf{E}_m + \underline{\mathbf{B}}_{km}\cdot\mathbf{H}_m],\ k=1,\,2,\,\dots,\,M\,,\tag{25a}$$

$$\mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\dots,M\}}[\underline{\mathbf{C}}_{km}\cdot\mathbf{E}_m + \underline{\mathbf{D}}_{km}\cdot\mathbf{H}_m],\ k=1,\,2,\,\dots,\,M\,.\tag{25b}$$

With $\underline{\mathbf{G}}_{km} = \underline{\mathbf{G}}(\mathbf{x}_k,\mathbf{x}_m)$ and $\underline{\mathbf{H}}_{km} = \underline{\mathbf{H}}(\mathbf{x}_k,\mathbf{x}_m)$, we have

$$\underline{\mathbf{A}}_{kk} = I + [k_0^2\underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k]\cdot(\underline{\mathbf{I}} - \underline{\varepsilon}_{r,k}) - i\omega\mu_0\underline{\mathbf{N}}_k\cdot\underline{\zeta}_{r,k}\,,\tag{26a}$$

$$\underline{\mathbf{B}}_{kk} = -[k_0^2\underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k]\cdot\underline{\xi}_{r,k} + i\omega\mu_0\underline{\mathbf{N}}_k\cdot(\underline{\mathbf{I}} - \underline{\mu}_{r,k})\,,\tag{26b}$$

$$\underline{\mathbf{C}}_{kk} = -[k_0^2\underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k]\cdot\underline{\zeta}_{r,k} - i\omega\varepsilon_0\underline{\mathbf{N}}_k\cdot(\underline{\mathbf{I}} - \underline{\varepsilon}_{r,k})\,,\tag{26c}$$

$$\underline{\mathbf{D}}_{kk} = I + [k_0^2\underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k]\cdot(\underline{\mathbf{I}} - \underline{\mu}_{r,k}) - i\omega\varepsilon_0\underline{\mathbf{N}}_k\cdot\underline{\xi}_{r,k}\,;\tag{26d}$$

and, for $k \neq m$,

$$\underline{\mathbf{A}}_{km} = -i\omega\mu_0\nu_m[i\omega\varepsilon_0\underline{\mathbf{G}}_{km}\cdot(\underline{\mathbf{I}}-\underline{\varepsilon}_{r,m})+\underline{\mathbf{H}}_{km}\cdot\underline{\zeta}_{r,m}]\,, \tag{27a}$$

$$\underline{\mathbf{B}}_{km} = i\omega\mu_0\nu_m[i\omega\varepsilon_0\underline{\mathbf{G}}_{km}\cdot\underline{\xi}_{r,m}+\underline{\mathbf{H}}_{km}\cdot(\underline{\mathbf{I}}-\underline{\mu}_{r,m})]\,, \tag{27b}$$

$$\underline{\mathbf{C}}_{km} = i\omega\varepsilon_0\nu_m[i\omega\mu_0\underline{\mathbf{G}}_{km}\cdot\underline{\zeta}_{r,m}-\underline{\mathbf{H}}_{km}\cdot(\underline{\mathbf{I}}-\underline{\varepsilon}_{r,m})]\,, \tag{27c}$$

$$\underline{\mathbf{D}}_{km} = -i\omega\varepsilon_0\nu_m[i\omega\mu_0\underline{\mathbf{G}}_{km}\cdot(\underline{\mathbf{I}}-\underline{\mu}_{r,m})-\underline{\mathbf{H}}_{km}\cdot\underline{\xi}_{r,m}]\,. \tag{27d}$$

In this straightforward version of the MOM, therefore, one may solve the algebraic equations (25a,b) for $\mathbf{E}_m$ and $\mathbf{H}_m$.

Once $\mathbf{E}_m$ and $\mathbf{H}_m$ have been calculated, $\mathbf{J}(\mathbf{x}_m)$ and $\mathbf{K}(\mathbf{x}_m)$ may be obtained using Eqs. (21a,b) and (23) for all $m \in \{1, 2, \ldots, M\}$, and the scattered fields in V_{ext} may then be computed as

$$\mathbf{E}_{\text{sca}}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}_{\text{inc}}(\mathbf{x}) = \sum_{m\in\{1,2,\ldots,M\}}\{\mu_m[i\omega\mu_0\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_m)\cdot\mathbf{J}(\mathbf{x}_m) - \underline{\mathbf{H}}(\mathbf{x},\mathbf{x}_m)\cdot\mathbf{K}(\mathbf{x}_m)]\},\quad \mathbf{x}\in V_{\text{ext}}\,, \tag{28a}$$

$$\mathbf{H}_{\text{sca}}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) - \mathbf{H}_{\text{inc}}(\mathbf{x}) = \sum_{m\in\{1,2,\ldots,M\}}\{\nu_m[i\omega\varepsilon_0\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}_m)\cdot\mathbf{K}(\mathbf{x}_m) + \underline{\mathbf{H}}(\mathbf{x},\mathbf{x}_m)\cdot\mathbf{J}(\mathbf{x}_m)]\},\quad \mathbf{x}\in V_{\text{ext}}\,, \tag{28b}$$

which follow from Eqs. (22a,b). Since all $\mathbf{x}_m$ are generally distributed around the origin, these expressions lead to

$$\text{Lim}_{k_0x_s\to\infty}\mathbf{E}_{\text{sca}}(x_s\mathbf{u}_s) \cong [\exp(ik_0x_s)/x_s]\mathbf{F}_{\text{sca}}(\mathbf{u}_s)\,, \tag{29a}$$

$$\text{Lim}_{k_0x_s\to\infty}\mathbf{H}_{\text{sca}}(x_s\mathbf{u}_s) \cong [\exp(ik_0x_s)/x_s](k_0/\omega\mu_0)\mathbf{u}_s\times\mathbf{F}_{\text{sca}}(\mathbf{u}_s)\,, \tag{29b}$$

in the far zone, with the far-zone scattering amplitude defined by

$$\mathbf{F}_{\text{sca}}(\mathbf{u}_s) = -i\mathbf{u}_s\times\sum_m\{\nu_m\exp(-ik_0\mathbf{u}_s\cdot\mathbf{x}_m)[\omega\mu_0\mathbf{u}_s\times\mathbf{J}(\mathbf{x}_m)+k_0\mathbf{K}(\mathbf{x}_m)]/4\pi\}\,. \tag{29c}$$

7. The Coupled Dipole Method

The heart of the MOM is constituted by Eqs. (24a,b) that involve the fields, $\mathbf{E}_k$ and $\mathbf{H}_k$, that are *actually present* at $\mathbf{x}_k$. However, in the coupled dipole method one considers the fields that *excite* the subregion V_k. To make the corresponding transformations, it is observed that the right-hand sides of Eqs. (24a,b) are fields that do not have their sources in V_k. In other words, the fields that excite V_k have to be

$$\mathbf{E}_{\text{exc},k} = \mathbf{E}_{\text{inc}}(\mathbf{x}_k) + \sum_{m,m\neq k}\nu_m\{i\omega\mu_0\underline{\mathbf{G}}_{km}\cdot\mathbf{J}(\mathbf{x}_m) - \underline{\mathbf{H}}_{km}\cdot\mathbf{K}(\mathbf{x}_m)\}\,, \tag{30a}$$

$$\mathbf{H}_{\mathrm{exc},k} = \mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) + \sum_{m,m\neq k} \nu_m \left\{ i\omega\varepsilon_0\nu \underline{\mathbf{G}}_{km} \cdot \mathbf{K}(\mathbf{x}_m) + \underline{\mathbf{H}}_{km} \cdot \mathbf{J}(\mathbf{x}_m) \right\} , \quad (30\text{b})$$

consistent with the long-wavelength approximation.

After making the identifications (30a,b), Eqs. (24a,b) are rewritten as

$$\mathbf{E}_k - i\omega\mu_0[\underline{\mathbf{M}}_k - (1/k_0^2)\underline{\mathbf{L}}_k] \cdot \mathbf{J}(\mathbf{x}_k) + \underline{\mathbf{N}}_k \cdot \mathbf{K}(\mathbf{x}_k) = \mathbf{E}_{\mathrm{exc},k}, \ k = 1, 2, \ldots, M , \quad (31\text{a})$$

$$\mathbf{H}_k - i\omega\varepsilon_0[\underline{\mathbf{M}}_k - (1/k_0^2)\underline{\mathbf{L}}_k] \cdot \mathbf{K}(\mathbf{x}_k) - \underline{\mathbf{N}}_k \cdot \mathbf{J}(\mathbf{x}_k) = \mathbf{H}_{\mathrm{exc},k}, \ k = 1, 2, \ldots, M . \quad (31\text{b})$$

But Eqs. (31a,b) are nothing but

$$\underline{\mathbf{A}}_{kk} \cdot \mathbf{E}_k + \underline{\mathbf{B}}_{kk} \cdot \mathbf{H}_k = \mathbf{E}_{\mathrm{exc},k}, \ k = 1, 2, \ldots, M , \quad (32\text{a})$$

$$\underline{\mathbf{C}}_{kk} \cdot \mathbf{E}_k + \underline{\mathbf{D}}_{kk} \cdot \mathbf{H}_k = \mathbf{H}_{\mathrm{exc},k}, \ k = 1, 2, \ldots, M ; \quad (32\text{b})$$

hence,

$$\mathbf{E}_k = \underline{\boldsymbol{\Delta}}_{e,k}^{-1} \cdot [\underline{\mathbf{B}}_{kk}^{-1} \cdot \mathbf{E}_{\mathrm{exc},k} - \underline{\mathbf{D}}_{kk}^{-1} \cdot \mathbf{H}_{\mathrm{exc},k}], \ k = 1, 2, \ldots, M , \quad (33\text{a})$$

$$\mathbf{H}_k = \underline{\boldsymbol{\Delta}}_{h,k}^{-1} \cdot [\underline{\mathbf{A}}_{kk}^{-1} \cdot \mathbf{E}_{\mathrm{exc},k} - \underline{\mathbf{C}}_{kk}^{-1} \cdot \mathbf{H}_{\mathrm{exc},k}], \ k = 1, 2, \ldots, M , \quad (33\text{b})$$

where

$$\underline{\boldsymbol{\Delta}}_{e,k} = \underline{\mathbf{B}}_{kk}^{-1} \cdot \underline{\mathbf{A}}_{kk} - \underline{\mathbf{D}}_{kk}^{-1} \cdot \underline{\mathbf{C}}_{kk}, \ \underline{\boldsymbol{\Delta}}_{h,k} = \underline{\mathbf{A}}_{kk}^{-1} \cdot \underline{\mathbf{B}}_{kk} - \underline{\mathbf{C}}_{kk}^{-1} \cdot \underline{\mathbf{D}}_{kk} . \quad (34\text{a,b})$$

Finally, the use of Eqs. (21a,b) and (23) yields

$$\mathbf{J}(\mathbf{x}_k)/i\omega\varepsilon_0 = (\{\underline{\mathbf{I}} - \underline{\boldsymbol{\varepsilon}}_{r,k}\} \cdot \underline{\boldsymbol{\Delta}}_{e,k}^{-1} \cdot \underline{\mathbf{B}}_{kk}^{-1} - \underline{\boldsymbol{\xi}}_{r,k} \cdot \underline{\boldsymbol{\Delta}}_{h,k}^{-1} \cdot \underline{\mathbf{A}}_{kk}^{-1}) \cdot \mathbf{E}_{\mathrm{exc},k}$$
$$+ (-\{\underline{\mathbf{I}} - \underline{\boldsymbol{\varepsilon}}_{r,k}\} \cdot \underline{\boldsymbol{\Delta}}_{e,k}^{-1} \cdot \underline{\mathbf{D}}_{kk}^{-1} + \underline{\boldsymbol{\xi}}_{r,k} \cdot \underline{\boldsymbol{\Delta}}_{h,k}^{-1} \cdot \underline{\mathbf{C}}_{kk}^{-1}) \cdot \mathbf{H}_{\mathrm{exc},k} , \quad (35\text{a})$$

$$\mathbf{K}(\mathbf{x}_k)/i\omega\mu_0 = (-\underline{\boldsymbol{\zeta}}_{r,k} \cdot \underline{\boldsymbol{\Delta}}_{e,k}^{-1} \cdot \underline{\mathbf{B}}_{kk}^{-1} + \{\underline{\mathbf{I}} - \underline{\boldsymbol{\mu}}_{r,k}\} \cdot \underline{\boldsymbol{\Delta}}_{h,k}^{-1} \cdot \underline{\mathbf{A}}_{kk}^{-1}) \cdot \mathbf{E}_{\mathrm{exc},k}$$
$$+ (\underline{\boldsymbol{\zeta}}_{r,k} \cdot \underline{\boldsymbol{\Delta}}_{e,k}^{-1} \cdot \underline{\mathbf{D}}_{kk}^{-1} - \{\underline{\mathbf{I}} - \underline{\boldsymbol{\mu}}_{r,k}\} \cdot \underline{\boldsymbol{\Delta}}_{h,k}^{-1} \cdot \underline{\mathbf{C}}_{kk}^{-1}) \cdot \mathbf{H}_{\mathrm{exc},k} . \quad (35\text{b})$$

Consistently with the long-wavelength approach, we define the equivalent dipole moments, $\mathbf{p}_k$ and $\mathbf{m}_k$ — of the electric and the magnetic types, respectively — located at $\mathbf{x}_k$; thus,[16,29]

$$\mathbf{p}_k = (i/w)\nu_k \mathbf{J}(\mathbf{x}_k), \quad \mathbf{m}_k = (i/\omega)\nu_k \mathbf{K}(\mathbf{x}_k), \ k = 1, 2, 3, \ldots, M . \quad (36\text{a,b})$$

Equations (35a,b) and (36a,b) yield

$$\mathbf{p}_k = \underline{\mathbf{a}}_{ee,k} \cdot \mathbf{E}_{\mathrm{exc},k} + \underline{\mathbf{a}}_{eh,k} \cdot \mathbf{H}_{\mathrm{exc},k}, \ \mathbf{m}_k = \underline{\mathbf{a}}_{he,k} \cdot \mathbf{E}_{\mathrm{exc},k} + \underline{\mathbf{a}}_{hh,k} \cdot \mathbf{H}_{\mathrm{exc},k} , \quad (37\text{a,b})$$

where

$$\underline{\mathbf{a}}_{ee,k} = -\nu_k \varepsilon_0 \left(\{\underline{\mathbf{I}} - \underline{\varepsilon}_{r,k}\} \cdot \underline{\Delta}_{e,k}^{-1} \cdot \underline{\mathbf{B}}_{kk}^{-1} - \underline{\xi}_{r,k} \cdot \underline{\Delta}_{h,k}^{-1} \cdot \underline{\mathbf{A}}_{kk}^{-1} \right) , \tag{38a}$$

$$\underline{\mathbf{a}}_{eh,k} = -\nu_k \varepsilon_0 \left(-\{\underline{\mathbf{I}} - \underline{\varepsilon}_{r,k}\} \cdot \underline{\Delta}_{e,k}^{-1} \cdot \underline{\mathbf{D}}_{kk}^{-1} + \underline{\xi}_{r,k} \cdot \underline{\Delta}_{h,k}^{-1} \cdot \underline{\mathbf{C}}_{kk}^{-1} \right) , \tag{38b}$$

$$\underline{\mathbf{a}}_{he,k} = -\nu_k \mu_0 \left(-\underline{\zeta}_{r,k} \cdot \underline{\Delta}_{e,k}^{-1} \cdot \underline{\mathbf{B}}_{kk}^{-1} + \{\underline{\mathbf{I}} - \underline{\mu}_{r,k}\} \cdot \underline{\Delta}_{h,k}^{-1} \cdot \underline{\mathbf{A}}_{kk}^{-1} \right) , \tag{38c}$$

$$\underline{\mathbf{a}}_{hh,k} = -\nu_k \mu_0 \left(\underline{\zeta}_{r,k} \cdot \underline{\Delta}_{e,k}^{-1} \cdot \underline{\mathbf{D}}_{kk}^{-1} - \{\underline{\mathbf{I}} - \underline{\mu}_{r,k}\} \cdot \underline{\Delta}_{h,k}^{-1} \cdot \underline{\mathbf{C}}_{kk}^{-1} \right) . \tag{38d}$$

These four dyadics $\underline{\mathbf{a}}_{ee,k}$, etc. are the polarizabilities of the material region V_k when it is immersed in free space all by itself.

With the definitions (36a,b), Eqs. (30a,b) become

$$\mathbf{E}_{\mathrm{exc},k} = \mathbf{E}_{\mathrm{inc}}(\mathbf{x}_k) - i\omega \sum_{m,m\neq k} \{ i\omega\mu_0 \underline{\mathbf{G}}_{km} \cdot \mathbf{p}_m - \underline{\mathbf{H}}_{km} \cdot \mathbf{m}_m \} , \tag{39a}$$

$$\mathbf{H}_{\mathrm{exc},k} = \mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) - i\omega \sum_{m,m\neq k} \{ i\omega\varepsilon_0 \underline{\mathbf{G}}_{km} \cdot \mathbf{m}_m + \underline{\mathbf{H}}_{km} \cdot \mathbf{p}_m \} , \tag{39b}$$

and the further substitution of Eqs. (37a,b) leads to

$$\begin{aligned} \mathbf{E}_{\mathrm{exc},k} = \mathbf{E}_{\mathrm{inc}}(\mathbf{x}_k) - i\omega \sum_{m,m\neq k} & \left[\{ i\omega\mu_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{ee,m} - \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{he,m} \} \cdot \mathbf{E}_{\mathrm{exc},m} \right. \\ & \left. + \{ i\omega\mu_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{eh,m} - \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{hh,m} \} \cdot \mathbf{H}_{\mathrm{exc},m} \right] , \end{aligned} \tag{40a}$$

$$\begin{aligned} \mathbf{H}_{\mathrm{exc},k} = \mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) - i\omega \sum_{m,m\neq k} & \left[\{ i\omega\varepsilon_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{hh,m} + \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{eh,m} \} \cdot \mathbf{H}_{\mathrm{exc},m} \right. \\ & \left. + \{ i\omega\varepsilon_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{he,m} + \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{ee,m} \} \cdot \mathbf{E}_{\mathrm{exc},m} \right] . \end{aligned} \tag{40b}$$

Equations (40a,b) constitute the core of the CDM and can be solved in terms of the 6M algebraic scalar equations

$$\mathbf{E}_{\mathrm{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\ldots,M\}} \left[\underline{\mathbf{P}}_{km} \cdot \mathbf{E}_{\mathrm{exc},m} + \underline{\mathbf{Q}}_{km} \cdot \mathbf{H}_{\mathrm{exc},m} \right] , \; k = 1, 2, \ldots, M , \tag{41a}$$

$$\mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\ldots,M\}} \left[\underline{\mathbf{R}}_{km} \cdot \mathbf{E}_{\mathrm{exc},m} \underline{\mathbf{S}}_{km} \cdot \mathbf{H}_{\mathrm{exc},m} \right] , \; k = 1, 2, \ldots, M , \tag{41b}$$

for the cartesian components of $\mathbf{E}_{\mathrm{exc},k}$ and $\mathbf{H}_{\mathrm{exc},k}$; here

$$\underline{\mathbf{P}}_{km} = \underline{\mathbf{I}}\delta_{km} + i\omega(i\omega\mu_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{ee,m} - \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{he,m})(1 - \delta_{km}) , \tag{42a}$$

$$\underline{\mathbf{Q}}_{km} = i\omega(i\omega\mu_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{eh,m} - \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{hh,m})(1 - \delta_{km}) , \tag{42b}$$

$$\underline{\mathbf{R}}_{km} = i\omega(i\omega\varepsilon_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{he,m} + \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{ee,m})(1 - \delta_{km}) , \tag{42c}$$

$$\underline{\mathbf{S}}_{km} = \underline{\mathbf{I}}\delta_{km} + i\omega(i\omega\varepsilon_0 \underline{\mathbf{G}}_{km} \cdot \underline{\mathbf{a}}_{hh,m} + \underline{\mathbf{H}}_{km} \cdot \underline{\mathbf{a}}_{eh,m})(1 - \delta_{km}) , \tag{42d}$$

δ_{km} being the Kronecker delta.

The solution of Eqs. (41a,b) can be used in Eqs. (37a,b) to find the dipole moments, $\mathbf{p}_m$ and $\mathbf{m}_m$, corresponding to all V_m, $m \in \{1, 2, \ldots, M\}$, and the scattered fields can then be ascertained from

$$\mathbf{E}_{\text{sca}}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}_{\text{inc}}(\mathbf{x}) = -i\omega \sum_{m\in\{1,2,\ldots,M\}} \{i\omega\mu_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_m)\cdot \mathbf{p}_m - \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}_m)\cdot \mathbf{m}_m\}, \ \mathbf{x} \in V_{\text{ext}}\,, \tag{43a}$$

and

$$\mathbf{H}_{\text{sca}}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) - \mathbf{H}_{\text{inc}}(\mathbf{x}) = -i\omega 5 \sum_{m\in\{1,2,\ldots,M\}} \{i\omega\varepsilon_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_m)\cdot \mathbf{m}_m + \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}_m)\cdot \mathbf{p}_m\}, \ \mathbf{x} \in V_{\text{ext}}\,. \tag{43b}$$

Equations (29a,b) still apply for the far zone scattered fields, but the far-zone scattering amplitude is now given by

$$\mathbf{F}_{\text{sca}}(\mathbf{u}_s) = -\omega \mathbf{u}_s \times \sum_m \{\exp(-ik_0\mathbf{u}_s\cdot\mathbf{x}_m)[\omega\mu_0\mathbf{u}_s \times \mathbf{p}_m + k_0\mathbf{m}_m]/4\pi\}\,. \tag{44}$$

8. Remarks on the Strong and the Weak Forms

Before continuing further, it is noted that Secs. 6 and 7 are refinements on Lakhtakia[16] who did not consider the self-terms arising from the dyadics $\underline{\mathbf{M}}_k$ and $\underline{\mathbf{N}}_k$, but confined himself to the self-terms arising out of $\underline{\mathbf{L}}_k$ only. With that statement as the backdrop, we may think of two forms of CDM:

(i) the Weak-CDM (W-CDM), in which the dyadics $\underline{\mathbf{M}}_k$ and $\underline{\mathbf{N}}_k$ are ignored, and

(ii) the Strong-CDM (S-CDM), in which the dyadics $\underline{\mathbf{M}}_k$ and $\underline{\mathbf{N}}_k$ are retained.

For isotropic dielectric scatterers, the W-CDM is exemplified by Purcell and Pennypacker[14] using isotropic dielectric spherical subregions, and was considerably generalized by Lakhtakia[16] for bianisotropic nonspherical subregions. The S-CDM does not appear to have been explicitly given elsewhere; in Sec. 7 is, perhaps, the first derivation of S-CDM.

We may also think of two forms of MOM:

(i) the Weak-MOM (W-MOM), in which the dyadics $\underline{\mathbf{M}}_k$ and $\underline{\mathbf{N}}_k$ are ignored, and

(ii) the Strong-MOM (S-MOM), in which the dyadics $\underline{\mathbf{M}}_k$ and $\underline{\mathbf{N}}_k$ are retained.

The W-MOM was derived for bianisotropic scatterers by Lakhtakia.[16] The S-MOM for bianisotropic scatterers appears to have been given explicitly for the first time in Sec. 6, though it has been used for isotropic dielectric scatterers for many years; see, e.g. Refs. 12, 33–35.

The W-MOM corresponds exactly to the W-CDM, as has been demonstrated by Lakhtakia,[15,16] while it follows from the two previous sections that the S-MOM corresponds exactly to the S-CDM. When all volumetric capacities ν_k are very small, the S-MOM/S-CDM effectively transmutes into the W-MOM/W-CDM. Generally stated, therefore, it follows that the scattering region $V_{\rm int}$ must be discretized into a larger number of subregions V_k when the W-MOM/W-CDM is used than if the S-MOM/S-CDM is used. Comparison of S-MOM results with the W-CDM results, with identical discretization of the scattering region — as was done, for instance, by Hage and Greenberg[17] — is not quite *kosher*.

9. Scattering and Absorption

Because $\mathbf{F}_{\rm sca}(\mathbf{u}_s)$ is of the form $\mathbf{u}_s \times \mathbf{b}$, it follows that $\mathbf{u}_s \cdot \mathbf{F}_{\rm sca}(\mathbf{u}_s) \equiv 0$; in turn, by virtue of Eqs. (29a,b), this implies that the scattered field in the far zone is transverse-electromagnetic[36] in character. This leads quite naturally to the planewave scattering dyadics, to which concept mere allusion suffices here.[37]

The time-averaged scattered power per unit solid angle

$$dP_{\rm sca}(\mathbf{u}_s)/d\Omega(\mathbf{u}_s) = (1/2)\,\mathrm{Real}\,[x_s^2 \mathbf{u}_s \cdot \{\mathbf{E}_{\rm sca}(x_s\mathbf{u}_s) \times \mathbf{H}^*_{\rm sca}(x_s\mathbf{u}_s)\}] \qquad (45a)$$

computed in the far-zone, with $d\Omega(\mathbf{u}_s) \equiv \sin\theta_s d\theta_s d\varphi_s$ as is customary in spherical coordinates, and the asterisk denoting the complex conjugate. From Eqs. (29a,b), therefore, we get

$$dP_{\rm sca}(\mathbf{u}_s)/d\Omega(\mathbf{u}_s) = (1/2\eta_0)\mathbf{F}_{\rm sca}(\mathbf{u}_s) \cdot \mathbf{F}^*_{\rm sca}(\mathbf{u}_s)\,, \qquad (45b)$$

where $\eta_0 = \sqrt{(\mu_0/\varepsilon_0)}$ is the intrinsic impedance of free space. Consequently, the time-averaged scattered power can be computed as

$$P_{\rm sca} = (1/2\eta_0)\int_0^{2\pi} d\varphi_s \int_0^{\pi} d\theta_s\, \sin\theta_s \mathbf{F}_{\rm sca}(\mathbf{u}_s) \cdot \mathbf{F}^*_{\rm sca}(\mathbf{u}_s)\,. \qquad (46)$$

Unless the scatterer material is intrinsically lossless,[21,38] there is absorption of electromagnetic energy in $V_{\rm int}$. The time-averaged power absorbed in $V_{\rm int}$ may be computed as the volume integral

$$P_{\rm abs} = \mathrm{Real}\left[(i\omega/2)\iiint_{V_{\rm int}} d^3x\,\{\mathbf{E}(\mathbf{x}) \cdot \mathbf{D}^*(\mathbf{x}) - \mathbf{H}^*(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x})\}\right]. \qquad (47)$$

Using Eqs. (3a,b) and (23), as well as the long-wavelength approximation of Sec. 6, this expression converts to the sum

$$P_{abs} = \text{Real}\left[(i\omega/2)\sum_m \nu_m \{\, \varepsilon_0 \mathbf{E}_m \cdot [\underline{\varepsilon}^*_{r,m} \cdot \mathbf{E}^*_m + \underline{\xi}^*_{r,m} \cdot \mathbf{H}^*_m] - \mu_0 \mathbf{H}^*_m \cdot [\underline{\zeta}_r \cdot \mathbf{E}_m + \underline{\mu}_r \cdot \mathbf{H}_m]\,\}\right]$$

$$= -(\omega/2)\,\text{Imag}\left[\sum_m \nu_m \{\, \varepsilon_0 \mathbf{E}_m \cdot [\underline{\varepsilon}^*_{r,m} \cdot \mathbf{E}^*_m + \underline{\xi}^*_{r,m} \cdot \mathbf{H}^*_m\,] - \mu_0 \mathbf{H}^*_m \cdot [\underline{\zeta}_r \cdot \mathbf{E}_m + \underline{\mu}_r \cdot \mathbf{H}_m]\right]. \tag{48}$$

Insofar as the MOM is concerned, the solution $\{(\mathbf{E}_m, \mathbf{H}_m);\ m = 1, 2, \ldots, M\}$ of Eqs. (25a,b) may be directly substituted into Eq. (48) for the computation of P_{abs}. The calculation of P_{abs} in the CDM is only slightly more complicated: the exciting fields $\{(\mathbf{E}_{exc,m}, \mathbf{H}_{exc,m});\ m = 1, 2, \ldots, M\}$ obtained by solving Eqs. (41a,b) have to substituted into Eqs. (33a,b) to get $\{(\mathbf{E}_m, \mathbf{H}_m);\ m = 1, 2, \ldots, M\}$ for use in Eq. (48). The total time-averaged power extinguished is the sum

$$P_{ext} = P_{sca} + P_{abs}\,. \tag{49}$$

Quite often, one is interested in the extinction of the plane wave

$$\mathbf{E}_{inc}(\mathbf{x}) = \mathbf{e}_{inc} \exp[ik_0 \mathbf{k}_{inc} \cdot \mathbf{x}],\ \mathbf{H}_{inc}(\mathbf{x}) = (1/\eta_0)\mathbf{k}_{inc} \times \mathbf{e}_{inc} \exp[ik_0 \mathbf{k}_{inc} \cdot \mathbf{x}]\,, \tag{50a,b}$$

where $\mathbf{e}_{inc}$ carries the units of volts per meter and $\mathbf{k}_{inc}$ is a dimensionless unit vector such that $\mathbf{e}_{inc} \cdot \mathbf{k}_{inc} = 0$. In this case, the total power extinguished by the presence of matter in V_{int} can be estimated using the forward scattering amplitude as[37]

$$P_{ext} = (2\pi/\omega\mu_0)\,\text{Imag}\,[\mathbf{e}^*_{inc} \cdot \mathbf{F}_{sca}(\mathbf{k}_{inc})]\,. \tag{51a}$$

Substitution of (44) and (50a) in (51a) then gives

$$P_{ext} = (k_0/2\mu_0)\,\text{Imag}\left[\sum_m \{\mathbf{E}^*_{inc}(\mathbf{x}_m) \cdot [\eta_0 \mathbf{p}_m - \mathbf{k}_{inc} \times \mathbf{m}_m]/4\pi\}\right] \tag{51b}$$

for use with the CDM. Comparison of Eqs. (29c) and (44) shows that the two formulae are identical if the identifications (36a,b) are borne in mind; hence,

$$P_{ext} = (1/2\eta_0)\,\text{Real}\left[\sum_m \nu_m \{\mathbf{E}^*_{inc}(\mathbf{x}_m) \cdot [\eta_0 \mathbf{J}(\mathbf{x}_m) - \mathbf{k}_{inc} \times \mathbf{K}(\mathbf{x}_m)]/4\pi\}\right] \tag{51c}$$

for MOM-users.

10. Isotropic Dielectric Scatterers

The scatterers most commonly studied by far are isotropic dielectric. Therefore, in this section, the MOM and the CDM are specialized to the case

$$\underline{\epsilon}_r(\mathbf{x}) = \epsilon_r(\mathbf{x})\underline{\mathbf{I}},\ \underline{\mu}_r(\mathbf{x}) = \underline{\mathbf{I}},\ \underline{\zeta}_r(\mathbf{x}) = \underline{\mathbf{0}},\ \underline{\xi}_r(\mathbf{x}) = \underline{\mathbf{0}},\ \mathbf{x} \in V_{\text{int}}\,, \tag{52a}$$

that implies

$$\mathbf{J}(\mathbf{x}) = i\omega\epsilon_0[\{1 - \epsilon_r(\mathbf{x})\}\mathbf{E}(\mathbf{x})],\ \mathbf{K}(\mathbf{x}) = \mathbf{0},\ \mathbf{x} \in V_{\text{int}}\,. \tag{52b}$$

Hence, we need to solve only the electric field integral equation

$$\mathbf{E}(\mathbf{x}) - \mathbf{E}_{\text{inc}}(\mathbf{x}) = \iiint_{V_{\text{int}}} d^3\mathbf{x}'\{i\omega\mu_0\underline{\mathbf{G}}(\mathbf{x},\mathbf{x}')\cdot\mathbf{J}(\mathbf{x}')\},\ \mathbf{x} \in V_{\text{int}} + V_{\text{ext}}\,. \tag{52c}$$

With the piecewise constant representation $\epsilon_r(\mathbf{x}) = \epsilon_{r,m}$ for $\forall\mathbf{x} \in V_m$, Eq. (52c) can be discretized as

$$\begin{aligned}\mathbf{E}_k &- i\omega\mu_0[\underline{\mathbf{M}}_k - (1/k_0^2)\underline{\mathbf{L}}_k]\cdot\mathbf{J}(\mathbf{x}_k)\\ &= \mathbf{E}_{\text{inc}}(\mathbf{x}_k) + \sum_{m,m\neq k} \nu_m\{i\omega\mu_0\underline{\mathbf{G}}(\mathbf{x}_k,\mathbf{x}_m)\cdot\mathbf{J}(\mathbf{x}_m),\ k = 1,2,\ldots,M\,,\end{aligned} \tag{53}$$

that leads to the MOM algebraic equations

$$\mathbf{E}_{\text{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\ldots,M\}} [\underline{\mathbf{A}}_{km}\cdot\mathbf{E}_m],\ k = 1,2,\ldots,M\,, \tag{54a}$$

with

$$\underline{\mathbf{A}}_{km} = \{\underline{\mathbf{I}} + [k_0^2\underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k](1-\epsilon_{r,k})\}\delta_{km} + \{\nu_m k_0^2(1-\epsilon_{r,m})\underline{\mathbf{G}}_{km}\}(1-\delta_{km})\,. \tag{54b}$$

Coming to CDM for isotropic dielectric scatterers, it is noted that

$$\underline{\mathbf{A}}_{kk}\cdot\mathbf{E}_k = \mathbf{E}_{\text{exc},k} \Rightarrow \mathbf{E}_k = \underline{\mathbf{A}}_{kk}^{-1}\cdot\mathbf{E}_{\text{exc},k}\,, \tag{55a}$$

whence

$$\mathbf{J}(\mathbf{x}_k)/i\omega\epsilon_0 = (1-\epsilon_{r,k})\underline{\mathbf{A}}_{kk}^{-1}\cdot\mathbf{E}_{\text{exc},k} \Rightarrow \mathbf{p}_k = -\nu_k\epsilon_0(1-\epsilon_{r,k})\underline{\mathbf{A}}_{kk}^{-1}\cdot\mathbf{E}_{\text{exc},k}\,, \tag{55b}$$

for $k = 1, 2, \ldots, M$. Hence,

$$\underline{\mathbf{a}}_{ee,k} = -\nu_k\epsilon_0(1-\epsilon_{r,k})\underline{\mathbf{A}}_{kk}^{-1},\ k = 1,2,\ldots,M\,, \tag{55c}$$

is the only non-zero polarizability dyadic. This finally leads to the CDM algebraic equations

$$\mathbf{E}_{\text{inc}}(\mathbf{x}_k) = \sum_{m\in\{1,2,\ldots,M\}} [\underline{\mathbf{P}}_{km}\cdot\mathbf{E}_{\text{exc},m}],\ k = 1,2,\ldots,M\,, \tag{56a}$$

where

$$\underline{\mathbf{P}}_{km} = \underline{\mathbf{I}}\delta_{km} - \{\omega^2\mu_0\underline{\mathbf{G}}_{km}\cdot\underline{\mathbf{a}}_{ee,m}\}(1-\delta_{km}). \tag{56b}$$

Although spheroidal and ellipsoidal subregions have been used,[39,40] it is commonplace in literature to have cubical or spherical subregions. Cubes and spheres have the same depolarization dyadic $\underline{\mathbf{L}}$, and it is customary[12] to estimate the $\underline{\mathbf{M}}$ dyadic of a cube as that of an equivoluminal sphere. Without any particular loss of generality therefore, the subregions are taken to be spherical in the remainder of this section.

Let the subregion V_m be the sphere of radius a_m with its center at $\mathbf{x}_m$. As a result, the volumetric capacity $\nu_m = (4\pi/3)a_m^3$, the depolarization dyadic $\underline{\mathbf{L}}_m = (1/3)\underline{\mathbf{I}}$, and the dyadic $\underline{\mathbf{M}}_m = (2/3k_0^2)[(1-ik_0a_m)\exp(ik_0a_m)-1]\underline{\mathbf{I}}$; see Fikioris[26] and Yaghjian.[28] (The dyadic $\underline{\mathbf{N}}$ for a sphere is always null, due to the asymmetry factor $\mathbf{u}_X \times \underline{\mathbf{I}}$ in Eq. (15); however, it is not needed in the present instance.) From Eq. (54b), the MOM self-term reduces to

$$\underline{\mathbf{A}}_{kk} = \{1+(1-\varepsilon_{r,k})[(2/3)(1-ik_0a_k)\exp(ik_0a_k)-1]\}\underline{\mathbf{I}}. \tag{57}$$

It is of interest to rewrite

$$\underline{\mathbf{A}}_{kk} = \{\acute{A}_{kk} + \mathring{A}_{kk}\}\underline{\mathbf{I}}, \tag{58a}$$

where

$$\acute{A}_{kk} = (\varepsilon_{r,k}+2)/3, \tag{58b}$$

$$\mathring{A}_{kk} = (2/3)(1-\varepsilon_{r,k})[(1-ik_0a_k)\exp(ik_0a_k)-1]. \tag{58c}$$

Both $\acute{A}_{kk}$ and $\mathring{A}_{kk}$ should be called self-terms; instead, perhaps mistakenly, only $\mathring{A}_{kk}$ has been accorded that honor in the literature.[17,18,34]

Going on to the self-term in CDM, it is noted the polarizability dyadic

$$\underline{\mathbf{a}}_{ee,k} = \underline{\mathbf{I}}(4\pi/3)a_k^3\varepsilon_0(\varepsilon_{r,k}-1)/(\acute{A}_{kk}+\mathring{A}_{kk}), \tag{59a}$$

may be rewritten as $\underline{\mathbf{a}}_{ee,k} = \underline{\mathbf{I}}a_{ee,k}$, where

$$a_{ee,k} = \alpha_{ee,k}/(1+\mathring{A}_{kk}/\acute{A}_{kk}), \tag{59b}$$

and

$$\alpha_{ee,k} = 4\pi a_k^3\varepsilon_0(\varepsilon_{r,k}-1)/(\varepsilon_{r,k}+2) \tag{59c}$$

is the polarizability of an electrically small dielectric sphere derivable from the Clausius–Mossotti relation.[41] Let $k_0a_k < 1$ in Eq. (58c) and $\mathring{A}_{kk}$ be evaluated correct to order $k_0^3a_k^3$, so that

$$a_{ee,k} \cong \alpha_{ee,k}/\{1-k_0^2(a_k^{-1}+2ik_0/3)\ alpha_{ee,k}/4\pi\varepsilon_0\}\ ; \tag{60}$$

it is observed here that the $(2ik_0^3/3)\alpha_{ee,k}/4\pi\varepsilon_0$ term in the denominator of the right hand side of (60) is the radiative reaction term of Draine.[19] More commonly, $\mathring{A}_{kk}$ is evaluated correct only to order k_0a_k, leading to $a_{ee,k} \cong \alpha_{ee,k}$, and thereby giving rise to the semi-microscopic flavor of this numerical approach.[15]

A comparison of the MOM and the CDM for isotropic dielectric scatterers is now in order. To facilitate such a comparison, it is reiterated that

$$\underline{\mathbf{A}}_{mm} = \{1 + (1 - \varepsilon_{r,m}) \times [(2/3)(1 - ik_0a_m)\exp(ik_0a_m) - 1]\}\underline{\mathbf{I}}\,, \quad \text{[S-MOM]} \quad (61a)$$

$$\underline{\mathbf{A}}_{mm} = \{(\varepsilon_{r,m} + 2)/3\}\underline{\mathbf{I}}\,; \quad \text{[W-MOM]} \quad (61b)$$

correspondingly,

$$a_{ee,m} = (4\pi/3)a_m^3\varepsilon_0(\varepsilon_{r,m} - 1)/\{1 + (1 - \varepsilon_{r,m}) \cdot [(2/3)(1 - ik_0a_m)\exp(ik_0a_m) - 1]\}\,, \quad \text{[S-CDM]} \quad (62a)$$

$$a_{ee,m} = 4\pi a_m^3\varepsilon_0(\varepsilon_{r,m} - 1)/(\varepsilon_{r,m} + 2)\,. \quad \text{[W-CDM]} \quad (62b)$$

There are at least two more CDM algorithms available. Draine[19] used

$$a_{ee,m} \cong \alpha_{ee,m}/\{1 - (i/6\pi\varepsilon_0)k_0^3\alpha_{ee,m}\}\,; \quad \text{[D-CDM]} \quad (62c)$$

while Dungey and Bohren,[20] with inspiration from Doyle[42] and the Mie analysis,[6,41] used

$$a_{ee,m} = (i6\pi\varepsilon_0/k_0^3)\frac{[\varepsilon_{r,m}^{1/2}\Psi(k_ma_m)\partial\Psi(k_oa_m) - \Psi(k_oa_m)\partial\Psi(k_ma_m)]}{[\varepsilon_{r,m}^{1/2}\Psi(k_ma_m)\partial\zeta(k_0a_m) - \zeta(k_0a_m)\partial\Psi(k_ma_m)]}\,, \quad \text{[DB-CDM]} \quad (62d)$$

where $k_m = k_0\varepsilon_{r,m}^{1/2}$, $\Psi(\beta) = \beta^{-1}\sin(\beta) - \cos(\beta)$, $\partial\Psi(\beta) = d\Psi/d\beta$, $\zeta(\beta) = -(i\beta^{-1} + 1)\exp(i\beta)$, and $\partial\zeta(\beta) = d\zeta/d\beta$.

The remarks made in Sec. 8 still apply, and it is repeated here that W-CDM should not be compared with S-MOM. Draine[19] and Dungey and Bohren[20] concluded from their numerical investigations that D-CDM and DB-CDM, respectively, generally provide results superior to those from W-CDM, but this does not come as a surprise since the self-terms in W-CDM (or W-MOM) are estimated the least accurately. On the other hand, although it is difficult to provide general enough conclusions for the adequacy of either D-CDM or DB-CDM vis-a-vis that of the S-CDM/S-MOM, it is safe to state that any claims of superiority — based purely on the estimation of some gross parameter, such as the scattering cross section — are debatable.

To round up this discussion on isotropic dielectric scatterers, it is noted that the far zone scattered amplitude for the present case turns out from Eq. (44) to be

$$\mathbf{F}_{\rm sca}(\mathbf{u}_s) = -(\omega^2\mu_0/4\pi)\mathbf{u}_s \times \left[\mathbf{u}_s \times \left\{\sum_m \exp(-ik_0\mathbf{u}_s\cdot\mathbf{x}_m)\mathbf{p}_m\right\}\right] \tag{63a}$$

for the CDM, and

$$\mathbf{F}_{\rm sca}(\mathbf{u}_s) = -(i\omega\mu_0/4\pi)\mathbf{u}_s \times \left[\mathbf{u}_s \times \left\{\sum_m \exp(-ik_0\mathbf{u}_s\cdot\mathbf{x}_m)\nu_m\mathbf{J}(\mathbf{x}_m)\right\}\right] \tag{63b}$$

from Eq. (29c) for the MOM. Next, since $\mathbf{E}_m = \mathbf{E}_{{\rm exc},m}/(\acute{A}_{mm} + \mathring{A}_{mm})$ and $\mathbf{p}_m = a_{ee,m}\mathbf{E}_{{\rm exc},m}$, the time-averaged power absorbed works out from Eq. (48) to be

$$P_{\rm abs} = -(\omega\varepsilon_0/2)\,{\rm Imag}\left[\sum_m(\varepsilon^*_{r,m}\nu_m)|\mathbf{p}_m/a_{ee,m}(\acute{A}_{mm} + \mathring{A}_{mm})|^2\right]. \tag{64}$$

Equations (58b,c) for $\acute{A}_{mm}$ and $\mathring{A}_{mm}$, and Eqs. (62a–d) for the $a_{ee,m}$ in the four CDM algorithms, may be substituted in Eq. (64) to obtain various estimates of $P_{\rm abs}$. In particular,

$$P_{\rm abs} = -(\omega/2\varepsilon_0)\,{\rm Imag}\left[\sum_m(\varepsilon^*_{r,m}/\nu_m)|\mathbf{p}_m/(1-\varepsilon_{r,m})|^2\right] \tag{65a}$$

for the W-CDM as well as for S-CDM, whence

$$P_{\rm abs} = -(1/2\omega\varepsilon_0)\,{\rm Imag}\left[\sum_m(\varepsilon^*_{r,m}\nu_m)|\mathbf{J}(\mathbf{x}_m)/(1-\varepsilon_{r,m})|^2\right] \tag{65b}$$

for both the W-MOM and the S-MOM.

Acknowledgments

The author thanks Dr. K. K. Phua and Miss P. H. Tham of World Scientific for the invitation to write this review.

This article is dedicated to the irrepressible Lazarus Long, a character created by Robert A. Heinlein, the noted science fiction writer. Long once opined:[43] "Anyone who cannot cope with mathematics is not fully human. At best he is a tolerable subhuman who has learned to wear shoes, bathe, and not make messes in the house."

References

1. E. J. Post, *Formal Structure of Electromagnetics* (North-Holland, Amsterdam, 1962).
2. F. N. H. Robinson, *Macroscopic Electromagnetism* (Pergamon, Oxford, 1973).

3. W. V. T. Rusch and P. D. Potter, *Analysis of Reflector Antennas* (Academic, NY, 1970).
4. A. Lakhtakia (ed.), *Essays on the Formal Aspects of Electromagnetic Theory* (World Scientific, Singapore, 1992), in press.
5. J. J. Bowman, T. B. A. Senior and P. L. E. Uslenghi (eds.), *Electromagnetic and Acoustic Scattering by Simple Shapes* (North-Holland, Amsterdam, 1969).
6. V. V. Varadan, A. Lakhtakia and V. K. Varadan (eds.), *Field Representations and Introduction to Scattering* (North-Holland, Amsterdam, 1991).
7. H. Massoudi, C. H. Durney and C. C. Johnson, "Long-wavelength electromagnetic power absorption in ellipsoidal models of man and animals", *IEEE Trans. Microwave Theory Tech.* **25** (1977) 47–52.
8. G. I. Rowlandson and P. W. Barber, "Absorption of higher-frequency RF energy by biological models: Calculations based on geometrical optics", *Radio Science* **14 (6S)** (1977) 43–50.
9. V. V. Varadan, A. Lakhtakia and V. K. Varadan, "Comments on recent criticism of the T-matrix method", *J. Acoust. Soc. Amer.* **84** (1988) 2280–2284.
10. A. Lakhtakia, *Near-Field Scattering and Absorption by Lossy Dielectrics at Resonance Frequencies*, Ph.D. dissertation (University of Utah, Salt Lake City, 1983).
11. R. F. Harrington, *Field Computation by Moment Methods* (McGraw-Hill, NY, 1968).
12. D. E. Livesay and K. M. Chen, "Electromagnetic fields induced inside arbitrarily shaped biological bodies", *IEEE Trans. Microwave Theory Tech.* **22** (1974) 1273–1280.
13. J. J. H. Wang, *Generalized Moment Methods in Electromagnetics* (Wiley, NY, 1991).
14. E. M. Purcell and C. R. Pennypacker, "Scattering and absorption of light by non-spherical dielectric grains", *Astrophys. J.* **186** (1973) 705–714.
15. A. Lakhtakia, "Macroscopic theory of the coupled dipole approximation method", *Opt. Commun.* **79** (1990) 1–5.
16. A. Lakhtakia, "General theory of the Purcell–Pennypacker scattering approach, and its extension to bianisotropic scatterers", *Astrophys. J.* **394** (1992).
17. J. I. Hage and J. M. Greenberg, "A model for the optical property of porous grains", *Astrophys. J.* **361** (1990) 251–259.
18. J. C. Ku and K.-H. Shim, "A comparison of solutions for light scattering and absorption by agglomerated or arbitrarily-shaped particles", *J. Quant. Spectrosc. Radiat. Transfer* 1992, in press.
19. B. T. Draine, "The discrete-dipole approximation and its application to interstellar graphite grains", *Astrophys. J.* **333** (1988) 848–872.
20. C. E. Dungey and C. F. Bohren, "Light scattering by nonspherical particles: a refinement to the coupled-dipole method", *J. Opt. Soc. Amer.* **A8** (1991) 81–87.
21. B. D. H. Tellegen, "The gyrator, a new electric network element", *Phillips Res. Repts.* **3** (1948) 81–101.
22. H. C. Chen, *Theory of Electromagnetic Waves* (McGraw-Hill, NY, 1983).
23. P. S. Reese and A. Lakhtakia, "A periodic chiral arrangement of thin bianisotropic sheets: effective properties", *Optik* **86** (1990) 47–50.
24. O. D. Kellogg, *Foundations of Potential Theory* (Dover Press, NY, 1953), Section VI-2.
25. J. J. H. Wang, "A unified and consistent view on the singularity of the electric dyadic Green's function in the source region", *IEEE Trans. Antennas Propagat.* **30** (1982) 463–468.
26. J. G. Fikioris, "Electromagnetic field inside a current-carrying region", *J. Math. Phys.* **6** (1965) 1617–1620.
27. A. Lakhtakia, "Polarizability dyadics of small chiral ellipsoids", *Chem. Phys. Lett.* **174** (1990) 583–586.

28. A. D. Yaghjian, "Electric dyadic Green's functions in the source region", *Proc. IEEE* **68** (1980) 248–263.
29. A. Lakhtakia, "Polarizability dyadics of small bianisotropic spheres", *J. Phys. France* **51** (1990) 2235–2242.
30. W. R. Stone (ed.), *Radar Cross Sections of Complex Objects* (IEEE Press, NY, 1990).
31. E. K. Miller, L. Medgyesi-Mitschang and E. H. Newman (eds.), *Computational Electromagnetics: Frequency-Domain Method of Moments* (IEEE Press, NY, 1991).
32. C. Athanasiadis, "Low-frequency electromagnetic scattering for a multi-layered scatterer", *Quart. J. Mech. Appl. Math.* **44** (1991) 55–67.
33. G. W. Hohmann, "Three-dimensional induced polarization and electromagnetic modeling", *Geophysics* **40** (1975) 309–324.
34. G. H. Goedecke and S. G. O'Brien, "Scattering by irregular inhomogeneous particles via the digitized Green's function algorithm", *Appl. Opt.* **27** (1988) 2431–2438.
35. M. F. Iskander, H. Y. Chen and J. E. Penner, "Optical branching and absorption by branched chains of aerosols", *Appl. Opt.* **28** (1989) 3083–3091.
36. R. F. Harrington, *Time-Harmonic Electromagnetic Fields* (McGraw-Hill, NY, 1961).
37. J. van Bladel, *Electromagnetic Fields* (Hemisphere, Washington, DC, 1985), Chap. 8.
38. J. A. Kong, "Theorems of bianisotropic media", *Proc. IEEE* **60** (1972) 1036–1046.
39. S. B. Singham, "Intrinsic optical activity in light scattering from an arbitrary particle", *Chem. Phys. Lett.* **130** (1986) 139–144.
40. V. K. Varadan, A. Lakhtakia and V. V. Varadan, "Scattering by beaded helices: anisotropy and chirality", *J. Wave-Mater. Interact.* **2** (1987) 153–160.
41. C. F. Bohren and D. R. Huffman, *Absorption and Scattering of Light by Small Particles* (Wiley, NY, 1983).
42. W. T. Doyle, "Optical properties of a suspension of metal spheres", *Phys. Rev.* **B39** (1989) 9852–9858.
43. R. A. Heinlein and D. F. Vassallo, *The Notebooks of Lazarus Long* (Putnam, NY, 1978).

International Journal of Modern Physics C, Vol. 4, No. 3 (1993) 721–722

ERRATA

STRONG AND WEAK FORMS OF THE METHOD OF MOMENTS AND THE COUPLED DIPOLE METHOD FOR SCATTERING OF TIME-HARMONIC ELECTROMAGNETIC FIELDS

AKHLESH LAKHTAKIA

[*Int. J. Mod. Phys.* **C3**, 583 (1992)]

The correct forms of several misprinted equations are as follows:

$$\underline{\mathbf{A}}_{kk} = \underline{\mathbf{I}} + [k_0^2 \underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k] \cdot (\underline{\mathbf{I}} - \underline{\varepsilon}_{r,k}) - i\omega\mu_0 \underline{\mathbf{N}}_k \cdot \underline{\zeta}_{r,k} \tag{26a}$$

$$\underline{\mathbf{D}}_{kk} = \underline{\mathbf{I}} + [k_0^2 \underline{\mathbf{M}}_k - \underline{\mathbf{L}}_k] \cdot (\underline{\mathbf{I}} - \underline{\mu}_{r,k}) - i\omega\varepsilon_0 \underline{\mathbf{N}}_k \cdot \underline{\xi}_{r,k} \tag{26d}$$

$$\mathbf{E}_{\mathrm{sca}}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - \mathbf{E}_{\mathrm{inc}}(\mathbf{x}) = \sum_{m \in \{1,2,\ldots,M\}} \{\nu_m [i\omega\mu_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_m) \cdot \mathbf{J}(\mathbf{x}_m) - \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}_m) \cdot \mathbf{K}(\mathbf{x}_m)]\}, \; \mathbf{x} \in V_{\mathrm{ext}} \tag{28a}$$

$$\mathbf{H}_{\mathrm{inc}}(\mathbf{x}_k) = \sum_{m \in \{1,2,\ldots,M\}} [\underline{\mathbf{R}}_{km} \cdot \mathbf{E}_{\mathrm{exc},m} \underline{\mathbf{S}}_{km} \cdot \mathbf{H}_{\mathrm{exc},m}], \; k = 1, 2, \ldots, M \tag{41b}$$

$$\mathbf{H}_{\mathrm{sca}}(\mathbf{x}) = \mathbf{H}(\mathbf{x}) - \mathbf{H}_{\mathrm{inc}}(\mathbf{x}) = -i\omega \sum_{m \in \{1,2,\ldots,M\}} \{i\omega\varepsilon_0 \underline{\mathbf{G}}(\mathbf{x}, \mathbf{x}_m) \cdot \mathbf{m}_m + \underline{\mathbf{H}}(\mathbf{x}, \mathbf{x}_m) \cdot \mathbf{p}_m\}, \; \mathbf{x} \in V_{\mathrm{ext}} \tag{43b}$$

$$P_{\rm abs} = {\rm Real}\left[(i\omega/2)\sum_m \nu_m \{\, \varepsilon_0 \mathbf{E}_m \cdot [\underline{\boldsymbol{\varepsilon}}^*_{r,m} \cdot \mathbf{E}^*_m + \underline{\boldsymbol{\xi}}^*_{r,m} \cdot \mathbf{H}^*_m] \right.$$

$$\left. - \mu_0 \mathbf{H}^*_m \cdot [\underline{\boldsymbol{\zeta}}_r \cdot \mathbf{E}_m + \underline{\boldsymbol{\mu}}_r \cdot \mathbf{H}_m] \,\}\right]$$

$$= -(\omega/2)\,{\rm Imag}\left[\sum_m \nu_m \{\, \varepsilon_0 \mathbf{E}_m \cdot [\underline{\boldsymbol{\varepsilon}}^*_{r,m} \cdot \mathbf{E}^*_m + \underline{\boldsymbol{\xi}}^*_{r,m} \cdot \mathbf{H}^*_m \,] \right.$$

$$\left. - \mu_0 \mathbf{H}^*_m \cdot [\underline{\boldsymbol{\zeta}}_r \cdot \mathbf{E}_m + \underline{\boldsymbol{\mu}}_r \cdot \mathbf{H}_m]\right] \tag{48}$$

$$P_{\rm ext} = (k_0/2\mu_0)\,{\rm Imag}\left[\sum_m \{\mathbf{E}^*_{\rm inc}(\mathbf{x}_m) \cdot [\eta_0 \mathbf{p}_m - \mathbf{k}_{\rm inc} \times \mathbf{m}_m]\}\right] \tag{51b}$$

$$P_{\rm ext} = (1/2\eta_0)\,{\rm Real}\left[\sum_m \nu_m \{\mathbf{E}^*_{\rm inc}(\mathbf{x}_m) \cdot [\eta_0 \mathbf{J}(\mathbf{x}_m) - \mathbf{k}_{\rm inc} \times \mathbf{K}(\mathbf{x}_m)]\}\right] \tag{51c}$$

$$a_{ee,k} = \alpha_{ee,k}/(1 + \mathring{A}_{kk}/\acute{A}_{kk}) \tag{59b}$$

$$a_{ee,k} \cong \alpha_{ee,k}/\{1 - k_0^2(a_k^{-1} + 2ik_0/3)\alpha_{ee,k}/4\pi\varepsilon_0\} \tag{60}$$

International
© World Sci

Ph

We dis
based
compa
which
allelize
the an
the on
we dis
three-d

Keywo

1. Intro

Relativist
equations
system is
constitue
free path
Such situ
relativisti
For ex
tational c
than cold
forces a s
propagate
termined
collisions,
sponding
quark-glu
three-dim