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VARIATIONS ON A PERSIAN THEME

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There lived in the fair city of Naishapur, modern Iran, a tentmaker (Khayyám) named Omar who was known far and wide for his mathematical skills. The 11-12 century genius later became better known, however, in the West for crafting the carousing quatrains collectively called *The Rubaiyat*. So famous is he now worldwide as a poet that his most marvelous algebraic structure is actually named after the 17th century French philosopher Blaise Pascal.

Pascal's triangle, or rather Khayyám's triangle, has fascinated myriads of people – mathematicians and others – and a contemporary summary of known results has been provided by Bondarenko [1]. In this article, a set of variations of this structure will be linked to the famous irrational $\sqrt{2}$ through an Italian connection: the 13th century Pisan savant Leonardo, also known as Fibonacci (blockhead). Another set of variations will be similarly linked to $\sqrt{(M+1)}$, where $(M+1) \geq 0$ is integral.

Khayyám's triangle can be viewed as a square array of integers $P_{n,m}$ that is generated using the following algorithm:

$$P_{n,0} = 1; \; n = 0, 1, 2, 3, \ldots, \tag{1a}$$

$$P_{0,m} = 1; \; m = 0, 1, 2, 3, \ldots, \tag{1b}$$

$$P_{n,m} = P_{n,m-1} + P_{n-1,m}; \; n > 0, m > 0; \tag{1c}$$

see Figure 1). An interesting aspect of this array are the anti-diagonal sums

$m \backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	2	3	4	5	6
2	1	3	6	10	15	21
3	1	4	10	20	35	56
4	1	5	15	35	70	126
5	1	6	21	56	126	252

Figure 1. The array $P_{n,m}$.

$m \backslash n$	0	1	2	3	4	5
0	M	1	1	1	1	1
1	1	$M+2$	$M+4$	$M+6$	$M+8$	$M+10$
2	1	$M+4$	$3M+10$	$5M+20$	$7M+34$	$9M+52$
3	1	$M+6$	$5M+20$	$13M+50$	$25M+104$	$41M+190$
4	1	$M+8$	$7M+34$	$25M+104$	$63M+258$	$129M+552$
5	1	$M+10$	$9M+52$	$41M+190$	$129M+552$	$321M+1362$

Figure 2. The array ${}^{M}C_{n,m}$.

$$D_N(P) = P_{n,0} + P_{N-1,1} + P_{n-2,2} + \ldots + P_{0,n} = 2^N; \; N \geq 0, \tag{2}$$

an identity that is very well known.

Set No. 1 of Variations

Consider now a square array of integers ${}^{M}C_{n,m}$ that is generated using the following variation of the algorithm (1):

$${}^{M}C_{0,0} = M, \tag{3a}$$

$${}^{M}C_{n,0} = 1; \; n = 1, 2, 3, \ldots, \tag{3b}$$

$${}^{M}C_{0,m} = 1; \; m = 1, 2, 3, \ldots, \tag{3c}$$

$${}^{M}C_{n,m} = {}^{M}C_{n-1,\,m-1} + {}^{M}C_{n,m-1} + {}^{M}C_{n-1,m}\,; \; n > 0, \, m > 0. \tag{3d}$$

Here M is integral, and can be positive, negative, or zero. The array has been sketched in Figure 2.

An investigation of the anti-diagonal sums

$$D_N({}^MC) = {}^MC_{n,0} + {}^MC_{N-1,1} + {}^MC_{n-2,2} + \ldots + {}^MC_{0,N}\,;\; N \geq 0 \tag{4}$$

of this array reveals an interesting sequence:

$$D_0({}^MC) = M, \quad D_1({}^MC) = 2 \tag{5a, b}$$

$$D_{N+1}({}^MC) = 2 \cdot D_N({}^MC) + D_{N-1}({}^MC), N > 1 \tag{5c}$$

It is easy to recognize that the $D_N({}^MC)$ form a generalized Fibonacci series [2]. Further, by repeated use of (5c), it can be observed that the infinite continued fraction

$$Q({}^MC) = 2 + \frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\cdots \tag{6}$$

is the limit

$$Q({}^MC) = \mathrm{Lim}_{N \to \infty}\,[\,D_{N+1}({}^MC)]/[D_N({}^MC)\,] \tag{7}$$

From (6), now it follows that

$$Q({}^MC) = 2 + \frac{1}{Q({}^MC)}\,, \tag{8}$$

whence

$$Q({}^MC) = 1 + \sqrt{2} \tag{9}$$

is independent of the integer M. Thus, not only is this set of variants of the gift of the tentmaker linked with the beautiful sequences of Fibonacci, a connection with the diagonal ($\sqrt{2}$) of a square with unit sides has also been obtained.

Set No. 2 of Variations

Consider now a square array of integers ${}^MS_{n,m}$ that are generated using the following variant of the algorithm (10):

$${}^MS_{n,0} = 1; n = 0, 1, 2, 3, \ldots, \tag{10a}$$

$${}^MS_{0,m} = 1; m = 0, 1, 2, 3, \ldots, \tag{10b}$$

$${}^MS_{n,m} = M \cdot {}^MS_{n-1,m-1} + {}^MS_{n,m-1} + {}^MS_{n-1,m}\;;\; n > 0,\; m > 0. \tag{10c}$$

Here, $(M + 1) \geq 0$ must be an integer, and the array is shown in Figure 3. It should be noted that ${}^{-1}S_{n,m} = 1$ for all n and m, while ${}^0S_{n,m} = P_{n,m}$.

An investigation of the anti-diagonal sums

$m \backslash n$	0	1	2	3	4	5
0	1	1	1	1	1	1
1	1	$M+2$	$2M+3$	$3M+4$	$4M+5$	$5M+6$
2	1	$2M+3$	M^2+6M+6	$3M^2+12M+10$	$6M^2+20M+15$	$10M^2+30M+21$
3	1	$3M+4$	$3M^2+12M+10$	$M^3+12M^2+30M+20$	$4M^3+30M^2+60M+35$	$10M^3+60M^2+105M+56$
4	1	$4M+5$	$6M^2+20M+15$	$4M^3+30M^2+60M+35$	$M^4+20M^3+90M^2+140M+70$	$5M^4+60M^3+210M^2+280M+126$
5	1	$5M+6$	$10M^2+30M+21$	$10M^3+60M^2+105M+56$	$5M^4+60M^3+210M^2+280M+126$	$M^5+30M^4+210M^3+560M^2+630M+252$

Figure 3. The array ${}^{M}S_{n,m}$.

$$D_N({}^MS) = {}^MS_{n,0} + {}^MS_{N-1,1} + {}^MS_{N-2,2} + \ldots + {}^MS_{0,N}\,; \; N \geq 0 \tag{11}$$

of this array reveals the generalized Fibonacci sequence:

$$D_0({}^MS) = 1, \qquad D_1({}^MS) = 2, \tag{12a, b}$$

$$D_{n+1}({}^MS) = 2 \cdot D_N({}^MS) + M \cdot D_{N-1}({}^MS), \; N > 1. \tag{12c}$$

Repeated use of (12c) leads to the infinite continued fraction

$$Q({}^MS) = 2 + \frac{M}{2+}\,\frac{M}{2+}\,\frac{M}{2+}\,\frac{M}{2+}\,\ldots \tag{13}$$

which is the limit

$$Q({}^MS) = \text{Lim}_{N\to\infty}\,[\,D_{N+1}\,({}^MS)\,]\,/\,[\,D_N({}^MS)\,]. \tag{14}$$

It follows from (13) that

$$Q({}^MS) = 2 + \frac{M}{Q({}^MS)}\,. \tag{15}$$

whence

$$Q({}^MS) = 1 + \sqrt{(M+1)}. \tag{16}$$

The second set of variants of Khayyám's triangle has thus been linked to the square roots of integers.

In conclusion, it is mentioned that the formula

$$D_N(^MS) = \left[\left\{ 1 + \sqrt{(M+1)} \right\}^{N+1} - \left\{ 1 - \sqrt{(M+1)} \right\}^{N+1} \right] / 2\sqrt{(M+1)}, \quad M \geq -1,\ N \geq 0, \qquad (17)$$

is easily verified by direct substitution in (12). It is to be noted that (17) resembles DeMoivre's formula for the usual Fibonacci series. The readers are invited to obtain a similar relation for $D_N(^M C)$.

References:

1. B. A. Bondarenko, *Generalized Pascal's Triangles and Pyramids; Fractals, Graphs andApplications*, Tashkent, Acad. Nauk Uzbekistan SSR, 1990.
2. M. R. Schroeder, *Number Theory in Science and Communication*, Berlin, Springer-Verlag, 1986.