TECHNICAL NOTE

Carpets and Rugs: An Exercise in Numbers

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ABSTRACT

An algorithm based on square arrays of positive integers, which was created for the purpose of constructing visually appealing designs for carpets and rugs, is described by the authors. A discussion of several examples follows.

God himself made the whole numbers: everything else is the work of man.

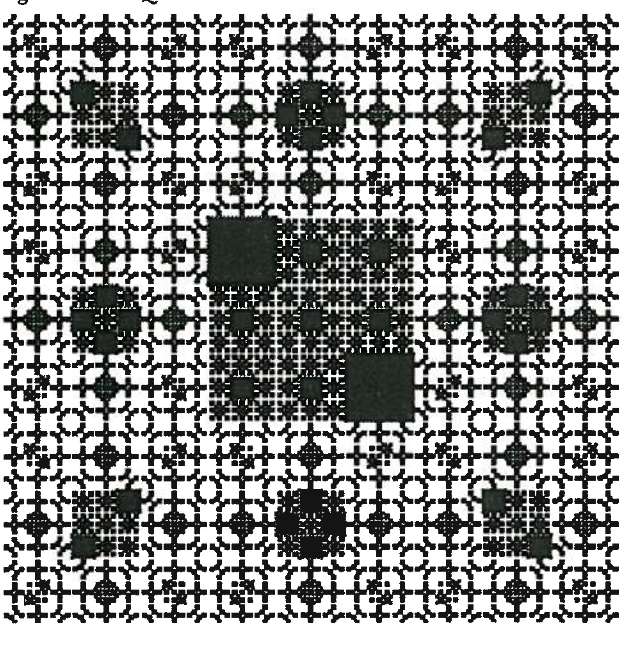
—Leopold Kronecker

Most, if not all, of us make sense of our respective environs by imposing our personal templates on our sensory perceptions [1]. The structure of the individual's template is determined partly by personal experiences and partly by heritage [2]. In the end, however, there is an underlying unity among all templates: though a Bokhara carpet is very different from a Navajo rug at first glance, both of these articles are patterned using the same geometrical principles.

Not only are numbers aesthetically appealing to the pure mathematician, they pervade all aspects of our lives. All of us know of mathematically illiterate persons who possess an almost uncanny ability to count. This is particularly exemplified by the handweavers of rugs and carpets. Over the centuries, families of weavers have perfected designs that have passed from one generation to the next. Every so often, a master weaver has created new designs without the use of a loom upon which to weave except that within his or her mind's eye. This skill implies not only an ability to count, but also the talent to arrange elemental patterns in some geometric fashion to achieve a fabulous effect. No wonder the Ionians declared that whole numbers are the elements of the universe.

In this article, we describe an experiment with square arrays of integers that was begun simply with the intention of studying some properties of prime numbers, but that has yielded spectacular patterns. In many ways these designs could not have been obtained by either of us if the computer revolution had not taken place. Therefore, we have still more cause to wonder at the capabilities of the human mind in general and those of master weavers in particular.

Fig. 1. $G = 9$ (cool series), Group 1 carpet design, inkjet print, 1990. This carpet design was produced through the authors' algorithm with $Q = 9 = 3^2$.



THE ALGORITHM

We generated an $N \times N$ array of integers $a(m,n)$ defined by the simple rule:

$$a(m,n) = a(m-1,n) + a(m,n-1) + a(m-1,n-1) \quad (1)$$

where $0 \le m \le N-1$ and $0 \le n \le N-1$. To initiate the process, we defined the first row of numbers $a(0,n)$ for all n, and set $a(n,0) = a(0,n)$. Once the entire array had been generated, the next step consisted of replacing $a(m,n)$ by the number p_{mn}, defined by

$$p_{mn} = a(m,n) \bmod Q \quad (2)$$

which is the remainder obtained after dividing $a(m,n)$ by the integer Q. The final step was to let $q_{mn} = 1$ if $p_{mn} = 0$, and $q_{mn} = 0$ if $p_{mn} \neq 0$, resulting in a distribution of 0's and 1's on a square lattice. Parenthetically, at a later stage we ascertained that the operation given together by equations (1) and (2) can be performed entirely in modular arithmetic, since equation (1) can be replaced by

$$a(m,n) = a(m-1,n) + 2\,\Sigma_{r=0,1,2,\ldots,n-1}\, a(m-1,r) \quad (3)$$

The color pertinent to the location (m,n) was chosen using q_{mn} as well as the sum

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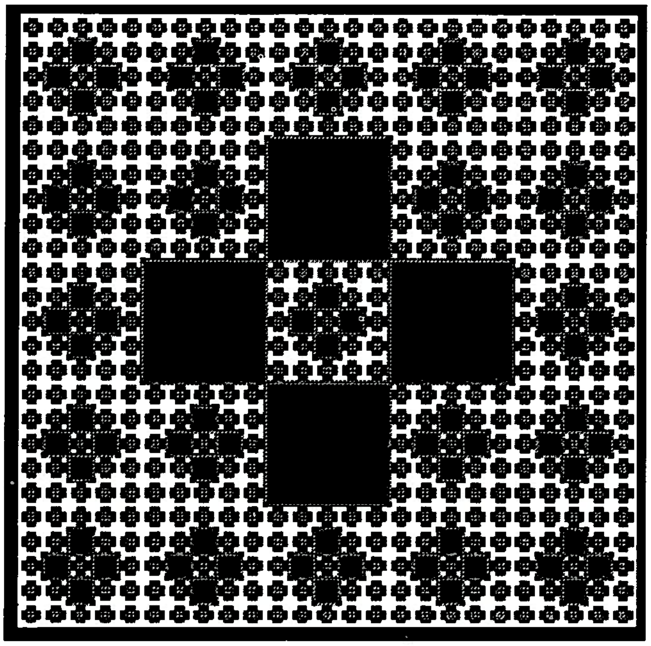

Fig. 2. $G = 5$ (cool series), Group 2 carpet design, inkjet print, 1990. This carpet design was produced through the authors' algorithm with $Q = 5$.

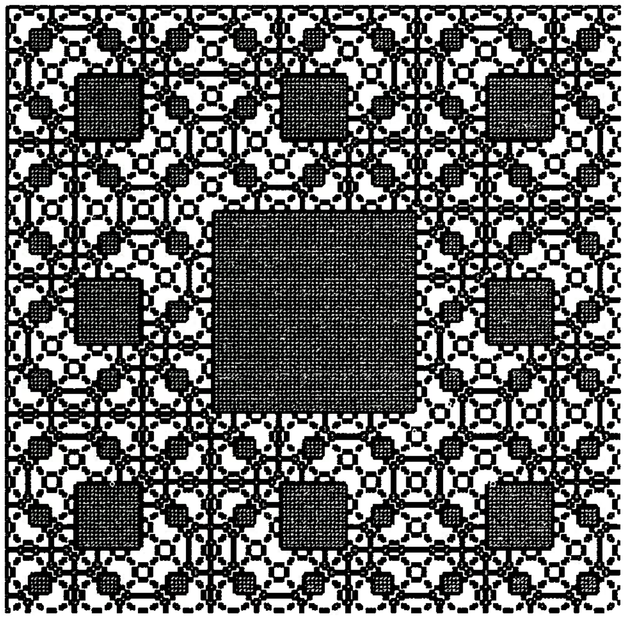

Fig. 3. $G = 6$ (warm series), Group 2 carpet design, inkjet print, 1990. This carpet design was produced through the authors' algorithm with $Q = 6 = 2 \times 3$.

$$s_{mn} = p_{mn} + p_{m-1,n} + p_{m,n-1} + p_{m-1,n-1} \quad (4)$$

while the palette for each carpet was determined by questioning several independent observers regarding the *harmoniousness* of the final product. The resulting palette, therefore, was not unique but did show some systematic trends that will be described later.

The Gallery of Carpets

Two separate groups of experiments were performed. In Group 1, later called the *cool series,* the elements $a\,(0,n)$ and $a\,(n,0)$ of the initial row and column were all set equal to 1, which resulted in all integers $a\,(m,n)$ being odd, as is evident from the recursion relation of equation (3). In Group 2 (later called the *warm series*), on the other hand, we set $a\,(0,n) = n \bmod 2$ and $a\,(n,0) = n \bmod 2$, so that the initial row and column were composed of alternating 0's and 1's, beginning with 0 at the left; again, it follows from equation (3) that odd and even numbers alternate in the succeeding rows of the array of integers $a\,(m,n)$. There is no point in having an even Q for Group 1 since in it all integers $a\,(m,n)$ are odd.

The gallery of carpets begins with Color Plate A No. 3 and Fig. 1, which belong to Group 1; the values of Q in these figures are, respectively, 5 and 9. It should be noted that the value of Q selected for Color Plate A No. 3 is a prime number. This has a particular result: visual inspection shows that the carpets of Group 1 for prime Q are self-similar [3]. In other words, the same motif is present at different scales. Due to limitations on the maximum value of N (here 625) that can be entertained for purposes of graphical display on our video screen, the self-similar nature could be best seen for $Q = 3$, 5 and 7. Furthermore, the special nature of using prime Q becomes clear when contrasting Color Plate A No. 3 with Fig. 1, which has $Q = 9$, a composite number. This does not mean that the carpet with $Q = 9$ or some other composite number is not visually attractive; the point is that the use of a prime Q gives a coherent self-similar structure to the carpets of Group 1.

Since Group 2 contains both odd and even integers $a\,(m,n)$, both odd and even Q's can be used; however, $Q = 2, 4, 8, 16, \ldots$, do not result in appealing designs, yielding merely orthogonal grids. Again, when Q is an odd prime, the resulting pattern is self-similar; this is exemplified in Fig. 2 for $Q = 5$. When Q is neither prime nor a power of 2, very exquisite designs result that, however, are not self-similar; these are exemplified in Fig. 3 for $Q = 6$.

Notes on Color

With the algorithmic description over, we now turn attention to the colors used for the production of the carpets of both Group 1 and Group 2. The algorithm was run and viewed on the monitor of a minicomputer with a 256-color palette. The carpet designs were printed on an inkjet printer with a more limited color palette. Although photographic slides of the terminal screen were taken with 35-mm color transparency film for chromaticity measurements, the images reproduced here as Color Plate A No. 3 and Figs 1–3 were originally produced as color inkjet prints.

As noted earlier, for ease of identification, carpets of Group 1 were constructed as a blue-green-yellow, or cool, series, and those of Group 2 were constructed as a red-brown-yellow, or warm, series. It was found that certain color schemes were more successful than others. The criteria used to determine color followed the geometry of the carpet; thus, the color of the carpet on a global scale was dependent upon the color derived on a local scale from geometry. Since the carpets with prime

Q exhibit self-similarity, we observed that those containing fine details required particular color schemes in order to appear harmonious.

The color harmonies that evolved in the course of our studies appeared to be related to interactions of colors [4] at the boundaries of the geometric subunits that constituted the overall structure of each carpet. Due to the dilatational invariance of coherent self-similar structures, the local color could not be divorced from the global color harmony.

The chromaticity coordinates of the individual colors resulting in the most pleasing carpets were studied using the standard method [5]. The average of five readings at 80 equally spaced wavelengths in the 3800–7800 Å range was recorded. At each wavelength, a background correction spectrum was measured on a black standard; it was then scaled and subtracted from the measured spectrum of the carpet. This corrected spectrum was then ratioed back to one measured for a white standard, thus eliminating the influence of the illuminating source. Standard observer curves were used to calculate the *u*, *v* and *w* coordinates in the Commission Internationale de l'Eclairage (CIE) 1976 chromaticity space.

Chromaticity measurements were made for the three colors used for the cool series and for the four colors used for the warm series. Both inkjet prints and color transparencies were utilized for this purpose. The results for the transparencies are shown in Fig. 4. These results are typical for the carpets judged 'harmonious' by several observers. Paths in the chromaticity space were found to be more or less along straight lines for both warm and cool series carpets.

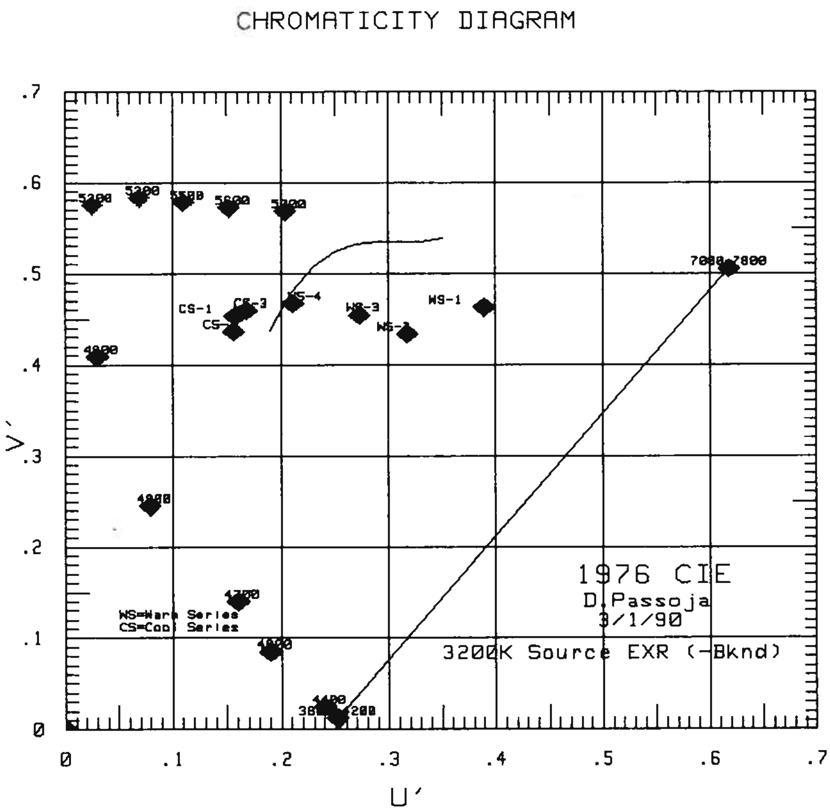

Fig. 4. Measured chromaticity diagram for cool and warm series carpet designs. The three colors used for the cool series (CS) and the four colors used for the warm series (WS) are identified.

References and Notes

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