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Short Note

Addition theorems for handed spherical vector wavefunctions

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Addition theorems for handed spherical vector wavefunctions. Translational and rotational addition theorems for handed spherical vector wavefunctions are given. It is shown that a left- (resp. right-) handed field appears as a left- (resp. right-) handed field, regardless of the placement and/or the orientation of the co-ordinate system utilized.

Additionstheoreme für händige sphärische Vektor-Wellenfunktionen. Translations- und Rotations-Additionstheoreme für händige sphärische Vektor-Wellenfunktionen werden abgeleitet. Es wird gezeigt, daß ein links- (resp. rechts-) händiges Feld als links- (resp. rechts-) händiges Feld erscheint, unabhängig von der Lage und/oder der Orientierung des verwendeten Koordinatensystems.

1. Spherical vector wavefunctions

Practitioners of electromagnetic theory generally prefer linearly-polarized fields. With respect to the (right-handed) co-ordinate system (r, θ, φ) centered at the origin O, vector wavefunctions $\boldsymbol{M}_{mn}$ and $\boldsymbol{N}_{mn}$ can be defined as [1]

$$\boldsymbol{M}_{mn}(k; r, \theta, \varphi) = h_n^{(1)}(kr) \exp[\mathrm{i}m\varphi] \cdot \{\boldsymbol{e}_\theta(\mathrm{i}m/\sin\theta)\, P_n^m - \boldsymbol{e}_\varphi \partial P_n^m\}, \tag{1a}$$

$$\boldsymbol{N}_{mn}(k; r, \theta, \varphi) = \boldsymbol{e}_r (kr)^{-1} h_n^{(1)}(kr) \exp[\mathrm{i}m\varphi]\, n(n+1) P_n^m + (kr)^{-1} [(\partial/\partial r)\, r h_n^{(1)}(kr)] \cdot \exp[\mathrm{i}m\varphi]\, \{\boldsymbol{e}_\varphi(\mathrm{i}m/\sin\theta)\, P_n^m + \boldsymbol{e}_\theta \partial P_n^m\}, \tag{1b}$$

in which the integer n varies from 1 to ∞; the integer m varies from $-n$ to $+n$; $\boldsymbol{e}_r$, $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\varphi$ are the unit vectors in the spherical co-ordinate system; $P_n^m \equiv P_n^m(\cos\theta)$ is the associated Legendre function and ∂P_n^m is its derivative with respect to θ; and $h_n^{(1)}(kr)$ is the spherical Hankel function of the first kind, k being the wavenumber. Both of these wavefunctions independently satisfy the vector Helmholtz equation, and are also divergenceless; each is also proportional to the circulation of the other one:

$$[\nabla^2 + k^2]\, \boldsymbol{M}_{mn} = 0 \quad \nabla \times \boldsymbol{M}_{mn} = k \boldsymbol{N}_{mn} \quad \nabla \cdot \boldsymbol{M}_{mn} = 0 \tag{2}$$

$$[\nabla^2 + k^2]\, \boldsymbol{N}_{mn} = 0 \quad \nabla \times \boldsymbol{N}_{mn} = k \boldsymbol{M}_{mn} \quad \nabla \cdot \boldsymbol{N}_{mn} = 0 \tag{3}$$

Translational and rotational theorems for these wavefunctions are very useful in multiple scattering problems [e.g., 2, 3], and have been investigated by several authors, e.g., Friedman and Russek [4], Stein [5] and Cruzan [6].

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Since it is not the objective of this communication to dwell on the addition theorems for the functions $\boldsymbol{M}_{mn}$ and $\boldsymbol{N}_{mn}$, it will suffice to state the following results.

Let a *rigid-body translation* of the coordinate system take place from origin O to another origin O', with (r', θ', φ') being the spatial variables in the primed co-ordinate system. Then, the translational addition theorems are given as [5]

$$\boldsymbol{M}_{\mu\nu}(k; r, \theta, \varphi) = \sum_{n=1,2,\ldots,\infty} \sum_{m=-n,\ldots,n} [C(\mu, \nu \mid m, n) \cdot \boldsymbol{M}_{mn}(k; r', \theta', \varphi') + D(\mu, \nu \mid m, n) \cdot \boldsymbol{N}_{mn}(k; r', \theta', \varphi')], \tag{4a}$$

$$\boldsymbol{N}_{\mu\nu}(k; r, \theta, \varphi) = \sum_{n=1,2,\ldots,\infty} \sum_{m=-n,\ldots,n} [C(\mu, \nu \mid m, n) \cdot \boldsymbol{N}_{mn}(k; r', \theta', \varphi') + D(\mu, \nu \mid m, n) \cdot \boldsymbol{M}_{mn}(k; r', \theta', \varphi')], \tag{4b}$$

in which the coefficients $C(\mu, \nu \mid m, n)$ and $D(\mu, \nu \mid m, n)$ are also dependent on the location of O' with respect to O.

Likewise, if a *rigid-body rotation* of the unprimed co-ordinate system takes place, the new system being (r, θ', φ'), then [5],

$$\boldsymbol{M}_{\mu n}(r, \theta, \varphi) = \sum_{m=-n,\ldots,n} \beta(\mu, m, n)\, \boldsymbol{M}_{mn}(r, \theta', \varphi'), \tag{5a}$$

$$\boldsymbol{N}_{\mu n}(r, \theta, \varphi) = \sum_{m=-n,\ldots,n} \beta(\mu, m, n)\, \boldsymbol{N}_{mn}(r, \theta', \varphi'), \tag{5b}$$

in which the coefficient $\beta(\mu, m, n)$ is given in terms of the Euler angles of the rotation.

Thus, a general addition theorem for both rotations and translations involves the transformation of $\boldsymbol{M}$ (or $\boldsymbol{N}$) into *both* $\boldsymbol{M}$ and $\boldsymbol{N}$ functions. The values of the coefficients $C(\mu, \nu \mid m, n)$, $D(\mu, \nu \mid m, n)$ and $\beta(\mu, m, n)$ are too complicated for inclusion in this communication; instead, the interested reader is referred to Stein [5].

Before carrying on, a specific characteristic of the $\boldsymbol{M}$ and $\boldsymbol{N}$ functions should be discussed. It should be noted that only the $\boldsymbol{N}$ functions contain radial components. On the basis of this significant difference between the two types of wavefunctions, it is quite common to decompose any electromagnetic field into two types of modes. The first mode is of the H-type in which the electric field is represented by the $\boldsymbol{M}$ functions and the magnetic field by the $\boldsymbol{N}$ functions: only the magnetic field has a radial component. In the E-type modes, the electric field is represented by the $\boldsymbol{N}$ functions and the magnetic field by the $\boldsymbol{M}$ functions: only the electric field has a radial component. Thus, the use of these functions, and of the consequent addition theorems (4a, b), gives rise to cross-polarization terms in multiple scattering [7].

2. Handed spherical vector wavefunctions

Rumsey [8] has suggested the use of circularly-polarized fields, instead of linearly-polarized fields, in some time-harmonic electromagnetic problems. His suggestion has considerable merit for certain radiation and propagation problems, and it has also been enthusiastically taken up by Baum [9] in connection with EMP simulation studies. Thus, following Bohren [10] *handed* spherical vector wavefunctions can be defined as per the relations

$$\boldsymbol{L}_{mn} = 2^{-1/2}(\boldsymbol{M}_{mn} + \boldsymbol{N}_{mn}); \quad \boldsymbol{R}_{mn} = 2^{-1/2}(\boldsymbol{M}_{mn} - \boldsymbol{N}_{mn}) \tag{6a, b}$$

in which $\boldsymbol{L}_{mn}$ and $\boldsymbol{R}_{mn}$, respectively, in the classical optics notation approximate to left- and right-circularly polarized plane waves for large kr in a given direction $\boldsymbol{e}_r$. These functions satisfy the vector Helmholtz equation, and are also divergenceless; each one is proportional to its own circulation:

$$\nabla \times \boldsymbol{L}_{mn} = k\boldsymbol{L}_{mn} \qquad \nabla \cdot \boldsymbol{L}_{mn} = 0 \tag{7}$$

$$\nabla \times \boldsymbol{R}_{mn} = -k\boldsymbol{R}_{mn} \qquad \nabla \cdot \boldsymbol{R}_{mn} = 0 \tag{8}$$

Properly speaking, as a result of the circulation equations that these functions satisfy, $\boldsymbol{L}$ and $\boldsymbol{R}$ are Beltrami functions [11, 12].

With the aid of the expressions (4a, b), it can be shown that the translational addition theorems for $\boldsymbol{L}$ and $\boldsymbol{R}$ functions are given as

$$\boldsymbol{L}_{\mu\nu}(k; r, \theta, \varphi) = \sum_{n=1,2,\ldots,\infty} \sum_{m=-n,\ldots,n} [C(\mu, \nu \mid m, n) + D(\mu, \nu \mid m, n)]\, \boldsymbol{L}_{mn}(k; r', \theta', \varphi'), \tag{9a}$$

$$\boldsymbol{R}_{\mu\nu}(k; r, \theta, \varphi) = \sum_{n=1,2,\ldots,\infty} \sum_{m=-n,\ldots,n} [C(\mu, \nu \mid m, n) - D(\mu, \nu \mid m, n)]\, \boldsymbol{R}_{mn}(k; r', \theta', \varphi'). \tag{9b}$$

In the same fashion, the rotational addition theorems can be found out to be

$$\boldsymbol{L}_{\mu n}(k; r, \theta, \varphi) = \sum_{m=-n,\ldots,n} \beta(\mu, m, n)\, \boldsymbol{L}_{mn}(k; r, \theta', \varphi'), \tag{10a}$$

$$\boldsymbol{R}_{\mu n}(k; r, \theta, \varphi) = \sum_{m=-n,\ldots,n} \beta(\mu, m, n)\, \boldsymbol{R}_{mn}(k; r, \theta', \varphi'). \tag{10b}$$

3. Discussion

This interesting aspect of the addition theorems (9a, b) and (10a, b) for the $\boldsymbol{L}$ and $\boldsymbol{R}$ functions should be noted: A left- (resp. right-) handed wavefunction is re-expressed in terms of only left- (resp. right-) handed wavefunctions, there being no *cross-handed* terms in either of these two sets of addition theorems. This suggests that interaction terms in scattering algorithms [7] involving more than one nonspherical scatterer can be handled more easily with $\boldsymbol{L}$ and $\boldsymbol{R}$ functions than with $\boldsymbol{M}$ and $\boldsymbol{N}$ functions. This comment is of particular significance for chiral-chiral composites [13] consisting of isotropic chiral inclusions dispersed in an isotropic chiral host medium in an isotropic chiral medium, the electromagnetic field must be expressed in terms of handed vector wavefunctions [14–16].

Another observation is as follows: a left- (resp. right-) handed field appears as a left- (resp. right-) handed field, regardless of the placement and/or the orientation of the co-ordinate system utilized; it is, of course, understood that all co-ordinate systems are right-handed. This is as it should be: handedness cannot be reversed by any combination of rigid body translations and rotations, but only by a mirror inversion!

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