Alternative Derivation of the Infinite-Medium Dyadic Green's Function for Isotropic Chiral Media

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Dyadic algebra and vector calculus have been utilized to obtain the infinite-medium dyadic Green's function for isotropic chiral media in terms of the Green's function for the scalar Helmholtz equation. The unified manner of the derivation is equally applicable to the one-, the two-, and the three-dimensional cases.

Eine andere Ableitung der dyadischen Greenschen Funktion für unendlich ausgedehnte isotrope chirale Materialien

Dyadische Algebra und Vektoranalyse eignen sich zur Ableitung der dyadischen Greenschen Funktion für unendlich ausgedehnte isotrope chirale Materialien auf der Basis der Greenschen Funktion für die skalare Helmholtz-Gleichung. Die einheitliche Berechnungsmethode läßt sich auch auf den ein-, zwei- und dreidimensionalen Fall anwenden.

Following the identification of natural optically active (chiral) media in nature [1] by Biot, Fresnel showed that their optical activity was due to the unequal phase velocities of left- and right-circularly polarized plane waves traversing them [2]. A differential equation for the electromagnetic field [1] in sourceless chiral media was given by MacCullagh [3] in 1836, and examined and confirmed by Cauchy by 1842 [4]. Plane wave propagation studies through natural optically active materials became very common thereafter, but a formal electromagnetic theory began developing only around the year 1900 [1], [2]. A landmark [5] in this context was the discovery of what is known now as the Bohren decomposition [6] that allowed the solution of many boundary value problems involving scatterers made of chiral media, and also lead to a first order characterization of the electromagnetic fields [7]. Further developments had to await the infinite-medium Green's function that is the focus of this communication.

Homogeneous, reciprocal, isotropic chiral media are characterized by the Drude-Born-Fedorov constitutive equations [6]

$$\boldsymbol{D} = \varepsilon[\boldsymbol{E} + \beta \nabla \times \boldsymbol{E}], \qquad (1\,\mathrm{a})$$

$$\boldsymbol{B} = \mu[\boldsymbol{H} + \beta \nabla \times \boldsymbol{H}], \qquad (1\,\mathrm{b})$$

in the frequency (ω) domain with ε and μ being the usual permittivity and permeability scalars, respectively, and β being the pseudoscalar chirality parameter. At a given frequency it is expected that $|k\beta| < 1$, where $k = \omega\sqrt{\varepsilon\mu}$ is not a wavenumber [6]. If $\boldsymbol{x} = \{x_1, x_2, x_3\}$ represents a point in 3-dimensional space, the infinite-medium, frequency-domain, dyadic Green's function $\mathfrak{G}(\boldsymbol{x})$ for chiral media is the solution of the inhomogeneous differential equation [6]

$$\operatorname{curl}\operatorname{curl} \mathfrak{G}(\boldsymbol{x}) - 2\gamma^2\beta \operatorname{curl} \mathfrak{G}(\boldsymbol{x}) - \gamma^2 \mathfrak{G}(\boldsymbol{x}) = \mathfrak{I}\delta(\boldsymbol{x}), \qquad (2)$$

where $\mathfrak{I}$ is the identity dyadic, $\delta(\boldsymbol{x})$ is the Dirac delta function and $\gamma^2 = k^2/(1 - k^2\beta^2)$. For later use, the quantities $\gamma_1 = k/(1 - k\beta)$ and $\gamma_2 = k/(1 + k\beta)$ should also be noted.

The three-dimensional solution of (2) was first given by Bassiri et al. [8] in 1986 by using spatial Fourier transforms. Despite the correct result, Weiglhofer [9] subsequently objected to the extraction of differential operators from Fourier integrals in this technique, undoubtedly a step lacking rigor; he went on to give a more rigorous derivation based on the algebra of differential operators [10]. Meanwhile, Lakhtakia et al. [11], [12] had reported one- and two-dimensional solution of (2) obtained using spatial Fourier transforms.

Since the reported solutions of (2) are correct [6, Chap. 13], the objective of this communication is simply to give another and a uniform way of solving (2). Neither Fourier transforms (as was done in [8] and [11]) nor operator algebra (as in [10]) will be used. Instead, the method of analysis will be focussed on using dyadic algebra and vector calculus to solve (2) from first principles in 3-dimensional space. Simplications for the case when dependence on x_3 and/or x_2 is neglected are concomitantly obtained [2].

We begin by attempting to find the solution $\mathfrak{G}_{\mathrm{h}}(\boldsymbol{x})$ of the homogeneous version of (2); in other words we look at

$$\operatorname{curl}\operatorname{curl} \mathfrak{G}_{\mathrm{h}}(\boldsymbol{x}) - 2\gamma^2\beta \operatorname{curl} \mathfrak{G}_{\mathrm{h}}(\boldsymbol{x}) - \gamma^2 \mathfrak{G}_{\mathrm{h}}(\boldsymbol{x}) = \mathbb{0}, \quad (3)$$

where $\mathbb{0}$ is the null dyadic. It is known that plane waves in chiral media are circularly polarized and transverse. Further it is reasonable to assume that the solution of (2) satisfies appropriate Sommerfeld radiation conditions. Therefore, the ansatz

$$\mathfrak{G}_{\mathrm{h}}(\boldsymbol{x}) = \mathfrak{I}u_{\mathrm{h}}(\boldsymbol{x}) + \operatorname{grad}\operatorname{grad} v_{\mathrm{h}}(\boldsymbol{x}) + \operatorname{curl}[\mathfrak{I}w_{\mathrm{h}}(\boldsymbol{x})] \quad (4)$$

is made where $u_{\mathrm{h}}(\boldsymbol{x})$, $v_{\mathrm{h}}(\boldsymbol{x})$ and $w_{\mathrm{h}}(\boldsymbol{x})$ are twice-differentiable scalar functions. Substituting (4) into (3) and employing standard vector-dyadic analysis [13] yields

$$\begin{aligned} &\nabla\nabla(u_{\mathrm{h}} - 2\gamma^2\beta w_{\mathrm{h}} - \gamma^2 v_{\mathrm{h}}) - \\ &\quad - \mathfrak{I}(\nabla^2 u_{\mathrm{h}} - 2\gamma^2\beta\nabla^2 w_{\mathrm{h}} + \gamma^2 u_{\mathrm{h}}) - \\ &\quad - \operatorname{curl}[\mathfrak{I}(\nabla^2 w_{\mathrm{h}} + 2\gamma^2\beta u_{\mathrm{h}} - \gamma^2 w_{\mathrm{h}})] = \mathbb{0} \quad (5)\end{aligned}$$

This differential equation will be satisfied for all $\boldsymbol{x}$ if the functions involved satisfy the three conditions

$$u_{\mathrm{h}}(\boldsymbol{x}) - 2\gamma^2\beta w_{\mathrm{h}}(\boldsymbol{x}) - \gamma^2 v_{\mathrm{h}}(\boldsymbol{x}) \equiv 0, \qquad (6\,\mathrm{a})$$

$$\nabla^2 u_{\mathrm{h}}(\boldsymbol{x}) - 2\gamma^2\beta\nabla^2 w_{\mathrm{h}}(\boldsymbol{x}) + \gamma^2 u_{\mathrm{h}}(\boldsymbol{x}) \equiv 0, \qquad (6\,\mathrm{b})$$

$$\nabla^2 w_{\mathrm{h}}(\boldsymbol{x}) + 2\gamma^2\beta u_{\mathrm{h}}(\boldsymbol{x}) + \gamma^2 w_{\mathrm{h}}(\boldsymbol{x}) \equiv 0. \qquad (6\,\mathrm{c})$$

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[1] Neither MacCullagh nor Cauchy used the term electromagnetic; the field actually considered them was the mechanical displacement of point masses on a lattice.

[2] Electromagnetic chirality is not possible in truly one- and two-dimensional worlds.

Eqs. (6b, c) may be put into the matrix form

$$\nabla^2 \begin{bmatrix} u_h(x) \\ w_h(x) \end{bmatrix} = -\gamma^2 \begin{bmatrix} 1+4\gamma^2\beta & 2\gamma^2\beta \\ 2\beta & 1 \end{bmatrix} \begin{bmatrix} u_h(x) \\ w_h(x) \end{bmatrix}; \quad (7)$$

but the matrix on the right hand side of (7) is diagonalizable [14], [15] and gives rise to the decomposition

$$u_h(x) = \gamma_1 h_1(x) + \gamma_2 h_2(x), \quad (8a)$$

$$w_h(x) = h_1(x) - h_2(x), \quad (8b)$$

where the scalar functions $h_1(x)$ and $h_2(x)$ must satisfy the homogeneous differential equations

$$[\nabla^2 + \gamma_1^2]\, h_1(x) = 0, \quad (9a)$$

$$[\nabla^2 + \gamma_2^2]\, h_2(x) = 0. \quad (9b)$$

Use of (9a, b) in the condition (6a) yields

$$v_h(x) = \gamma_1^{-1} h_1(x) + \gamma_2^{-1} h_2(x). \quad (10)$$

Thus the solution $\mathfrak{G}_h(x)$ of the homogeneous differential equation (3) can be obtained from the functions $h_1(x)$ and $h_2(x)$.

From comparing (2) and (3) it is clear that $\mathfrak{G}(x)$ and $\mathfrak{G}_h(x)$ must be quite similar in form. Therefore, the solution of (2) should have the form

$$\mathfrak{G}(x) = \alpha \{\mathfrak{I}[\gamma_1 g_1(x) + \gamma_2 g_2(x)] + \text{grad grad}\,[\gamma_1^{-1} g_1(x) + \gamma_2^{-1} g_2(x)] + \text{curl}\,(\mathfrak{I}[g_1(x) - g_2(x)])\}, \quad (11)$$

the properties of the scalar functions $g_1(x)$ and $g_2(x)$ yet to be determined, and α being some multiplicative constant also to be determined. Substitution of (11) into (2), followed by simple algebraic manipulations, leads to

$$-(\gamma_2 + \text{curl})\,[\mathfrak{I}\nabla^2 g_1 + \gamma_1^2 \mathfrak{I} g_1] - (\gamma_1 - \text{curl})\,[\mathfrak{I}\nabla^2 g_2 + \gamma_2^2 \mathfrak{I} g_2] = \alpha^{-1} \mathfrak{I}\delta(x). \quad (12)$$

This equation is identically satisfied if the functions $g_1(x)$ and $g_2(x)$, respectively, are the solutions of the differential equations

$$[\nabla^2 + \gamma_1^2]\, g_1(x) = -\delta(x), \quad (13a)$$

$$[\nabla^2 + \gamma_2^2]\, g_2(x) = -\delta(x), \quad (13b)$$

and the multiplicative constant

$$\alpha = k/2\gamma^2. \quad (13c)$$

The solution of (2) has thus been obtained as (11) in terms of α, $g_1(x)$ and $g_2(x)$.

Eqs. (13a, b) show that $g_1(x)$ and $g_2(x)$ are the well-known scalar Green's functions for the scalar Helmholtz equation. The general case is obtained as [16]

$$g_1(x) = \exp(i\gamma_1 \sqrt{x_1^2 + x_2^2 + x_3^2})/(4\pi \sqrt{x_1^2 + x_2^2 + x_3^2}), \quad (14a)$$

$$g_2(x) = \exp(i\gamma_2 \sqrt{x_1^2 + x_2^2 + x_3^2})/(4\pi \sqrt{x_1^2 + x_2^2 + x_3^2}), \quad (14b)$$

consistent with an $\exp(-i\omega t)$ time-dependence. Likewise, when $\partial/\partial x_3 \equiv 0$,

$$g_1(x) = (i/4)\, H_0^{(1)}(\gamma_1 \sqrt{x_1^2 + x_2^2}), \quad (15a)$$

$$g_2(x) = (i/4)\, H_0^{(1)}(\gamma_2 \sqrt{x_1^2 + x_2^2}), \quad (15b)$$

where $H_0^{(1)}(\cdot)$ is the cylindrical Hankel function of the first kind and order zero.

Finally, when $\partial/\partial x_3 \equiv 0$ and $\partial/\partial x_2 \equiv 0$,

$$g_1(x) = (i/2\gamma_1) \exp(i\gamma_1 |x_1|), \quad (16a)$$

$$g_2(x) = (i/2\gamma_2) \exp(i\gamma_2 |x_1|). \quad (16b)$$

In conclusion, using dyadic algebra and vector calculus, the infinite-medium dyadic Green's function for isotropic chiral media has been obtained in terms of the Green's function for the scalar Helmholtz equation. The unified manner of the derivation here yields the one-, the two-, and the three-dimensional cases in "one fell swoop", to quote Shakespeare in *Macbeth*.

References

[1] Applequist, J.: Optical activity: Biot's bequest. American Scientist **75** (1987), 59–67.

[2] Lakhtakia, A.: The editor's apology. In: Selected Papers on Natural Optical Activity, A. Lakhtakia, (ed.), Bellingham, Washington (USA): SPIE Optical Engg. Press, 1990.

[3] MacCullagh, J.: On the laws of the double refraction of quartz. Trans. Royal Irish Acad. **17** (1837), 461–469.

[4] Buchwald, J. Z.: Optics and the theory of the punctiform ether. Arch. Hist. Exact. Sci. **21** (1980), 245–278.

[5] Bohren, C. F.: Light scattering by an optically active sphere. Chem. Phys. Lett. **29** (1974), 458–462.

[6] Lakhtakia, A.; Varadan, V. K.; Varadan, V. V.: Time-harmonic electromagnetic fields in chiral media. Berlin: Springer, 1989.

[7] Lakhtakia, A.: First order characterization of electromagnetic fields in isotropic chiral media. AEÜ **44** (1990), 57–59.

[8] Bassiri, S.; Engheta, N.; Papas, C. H.: Dyadic Green's function and dipole radiation in chiral media. Alta Freq. **55** (1986), 83–88.

[9] Weiglhofer, W.: Isotropic chiral media and scalar Hertz potentials. J. Phys. A: Math. Gen. **21** (1988), 2249–2251.

[10] Weiglhofer, W.: A simple and straightforward derivation of the dyadic Green's function of an isotropic chiral medium. AEÜ **43** (1989), 51–52.

[11] Lakhtakia, A.; Varadan, V. V.; Varadan, V. K.: Field equations, Huygen's principle, integral equations, and theorems for radiation and scattering of electromagnetic waves in isotropic chiral media. J. Opt. Soc. Am. A **5** (1988), 175–184.

[12] Lakhtakia, A.; Varadan, V. K.; Varadan, V. V.: Comments on "One- and two-dimensional Green's functions in chiral media". IEEE Trans. AP-**38** (1990), 1514.

[13] van Bladel, J.: Electromagnetic fields. Washington, DC: Hemisphere, 1985.

[14] Hochstadt, H.: Differential equations: A modern approach. New York: Dover, 1975.

[15] Lakhtakia, A.; Varadan, V. K.; Varadan, V. V.: Propagation along the direction of inhomogeneity in an inhomogeneous chiral medium. Int. J. Engng. Sci. **27** (1989), 1267–1273.

[16] Morse, P. M.; Feshbach, H.: Methods of theoretical physics. New York: McGraw-Hill, 1953, Section 7.2.