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Reflection and transmission of normally-incident plane waves by a chiral slab with linear property variations

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Reflection and transmission of normally-incident plane waves by a chiral slab with linear property variations. Reflection by and transmission through a chiral slab are explored for normally incident plane waves, it being assumed that the constitutive parameters of the slab vary along the direction of incidence. It is shown that the Faraday and the Ampère-Maxwell equations decompose into two autonomous matrix differential equations. The effect of the chiral slab is accounted for by the matrices $\boldsymbol{S}^{\pm}$ that are evaluated for a linearly inhomogeneous chiral slab using two methods.

Reflektion und Transmission einer normal in eine chirale Platte mit veränderlichen linearen Materialeigenschaften einfallenden Planwelle. Reflexion an und Transmission durch eine chirale Platte werden für normal einfallende ebene Wellen untersucht. Es wird angenommen, daß sich die wirksamen Parameter der Platte längs der Einfallsrichtung ändern. Es wird gezeigt, daß die Faraday- und die Ampère-Maxwell-Gleichungen in zwei autonome Matrizen-Differential-Gleichungen aufspalten. Der Effekt der chiralen Platte wird durch die Matrizen $\boldsymbol{S}^{\pm}$ erklärt, welche für eine linear inhomogene chirale Platte auf zwei verschiedene Arten gewonnen werden können.

1. Introduction

Unidirectionally inhomogeneous media are commonly utilized for special purpose devices such as lenses and polarizers. Even for one-dimensional propagation in isotropic dielectric materials, analytic solutions are available only for linearly, quadratically or sinusoidally varying refractive index profiles. For these refractive index profiles, the solutions are obtained in terms of Airy, Weber and Mathieu functions [1–3], that are not easy to compute. The common course is to treat the unidirectionally medium as having piecewise constant properties and then employ variants of Jones calculus [e.g., 4–8]. So successful have such approaches been that a version called the multislice method is being used for the Reflection High Energy Electron Diffraction (RHEED) technique to examine thin and thick films [9, 10], as well as to design electron lenses [11, 12].

Potential uses of artificial chiral composite materials at suboptical frequencies have recently spurred interest in the electromagnetic wave theory for isotropic chiral media. Recent developments have been summarized in [13] to which work the interested reader is referred. The literature on inhomogeneous chiral materials is sparse. Weiglhofer [14, 15] has given a derivation of scalar potentials in a unidirectionally inhomogeneous chiral continuum. Propagation along the direction of inhomogeneity in a periodically inhomogeneous chiral medium has been considered using differential equations of the first [16] and the second [17] orders; off-axis propagation has also been examined [18]. Electromagnetic fields in a stratified chiral medium have been investigated using Jones calculus [18].

In this paper, reflection by and transmission through a chiral slab are explored for normally incident plane waves, it being assumed that the constitutive parameters of the slab vary along the direction of incidence. It is shown that the Faraday and the Ampère-Maxwell equations decompose into two autonomous matrix differential equations. The effect of the chiral slab is accounted for by two matrices $\underline{\underline{S}}^{\pm}$, that are evaluated for a linearly inhomogeneous chiral slab using two methods.

2. Preliminary analysis

Consider an inhomogeneous isotropic chiral slab of finite thickness ($0 \leqq z \leqq d$) interposed between two homogeneous isotropic achiral half spaces ($z \leqq 0$ and $z \geqq d$), the inhomogeneity of the slab being purely z-dependent. The constitutive equations can be compactly set up *everywhere* in the Drude-Born-Fedorov representation [13] as

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon(z)\,[\boldsymbol{E}(\boldsymbol{r}) + \beta(z)\,\nabla\times\boldsymbol{E}(\boldsymbol{r})] \tag{1a}$$

$$\boldsymbol{B}(\boldsymbol{r}) = \mu(z)\,[\boldsymbol{H}(\boldsymbol{r}) + \beta(z)\,\nabla\times\boldsymbol{H}(\boldsymbol{r})]\,. \tag{1b}$$

in which $\varepsilon(z)$ and $\mu(z)$ are the permittivity and the permeability scalars, respectively, and $\beta(z)$ is the chirality pseudoscalar. Use of (1 a, b) in the source-free Faraday and Ampère-Maxwell equations leads to

$$\nabla\times\boldsymbol{E}(\boldsymbol{r}) = i\omega\mu(z)\,[\boldsymbol{H}(\boldsymbol{r}) + \beta(z)\,\nabla\times\boldsymbol{H}(\boldsymbol{r})], \tag{2a}$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r}) = -\,i\omega\varepsilon(z)\,[\boldsymbol{E}(\boldsymbol{r}) + \beta(z)\,\nabla\times\boldsymbol{E}(\boldsymbol{r})], \tag{2b}$$

an $\exp[-i\omega t]$ time-dependence having been assumed.

Since only normal plane wave incidence on the slab is being considered, it follows that the electromagnetic field everywhere is dependent only on the z-coordinate. From (2a, b), this condition leads to

$$E_z(z) \equiv 0, \qquad H_z(z) \equiv 0, \quad |z| \leqq \infty\,. \tag{3}$$

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Further, $\partial/\partial x \equiv 0$ and $\partial/\partial y \equiv 0$ suggest the decompositions [16]

$$\boldsymbol{E}(z) = \boldsymbol{u}_+ E^+(z) + \boldsymbol{u}_- E^-(z), \tag{4a}$$

$$\boldsymbol{H}(z) = \boldsymbol{u}_+ H^+(z) + \boldsymbol{u}_- H^-(z), \tag{4b}$$

where

$$\boldsymbol{u}_\pm = [\boldsymbol{u}_x \pm i\boldsymbol{u}_y]/\sqrt{2}, \tag{4c}$$

$\boldsymbol{u}_x$ and $\boldsymbol{u}_y$ are unit cartesian vectors, while $E^\pm(z)$ and $H^\pm(z)$ are piecewise at least once-differentiable scalar functions. In the sequel it is also assumed that $\varepsilon(z)$, $\mu(z)$ and $\beta(z)$ are real, and that $\omega^2 \varepsilon(z)\,\mu(z)\,\beta^2(z) < 1$ for all $|z| \leqq \infty$.

Substitution of (4a–c) into (2a, b) yields two autonomous matrix equations

$$(d/dz)\,\underline{X}^\pm(z) = \pm\,\underline{\underline{B}}(z)\cdot \underline{X}^\pm(z), \tag{5a}$$

where

$$\underline{X}^\pm(z) = \text{column}\,[E^\pm(z), H^\pm(z)], \tag{5b}$$

is a 2-vector and

$$\underline{\underline{B}}(z) = [1 - \omega^2 \varepsilon(z)\,\mu(z)\,\beta^2(z)]^{-1} \cdot \begin{pmatrix} i\omega^2 \varepsilon(z)\,\mu(z)\,\beta(z) & -\omega\mu(z) \\ \omega\varepsilon(z) & i\omega^2\varepsilon(z)\,\mu(z)\,\beta(z) \end{pmatrix}. \tag{5c}$$

is a 2×2 matrix. As has been shown by Hochstadt [19], the solution of (5a) must be of the form

$$\underline{X}^\pm(z) = \underline{\underline{M}}^\pm(z-\xi)\cdot \underline{X}^\pm(\xi), \tag{6}$$

where the matrices $\underline{\underline{M}}^\pm(z)$, satisfying the differential equations $(d/dz)\,\underline{\underline{M}}^\pm(z) = \pm\,\underline{\underline{B}}(z)\cdot\underline{\underline{M}}^\pm(z)$, are called the *matrizants* of (6).

3. Reflection and transmission coefficients

Equation (6) results in a simple formulation of the normal reflection and transmission problem being solved here. In the region $z \leqq 0$, the homogeneous achiral medium is specified by

$$\varepsilon(z) = \varepsilon_a, \quad \mu(z) = \mu_a, \quad \beta(z) = 0, \quad z \leqq 0. \tag{7}$$

Without loss of generality, this region is taken to support both the incident and the reflected plane waves; thus [13],

$$\boldsymbol{E}(z) = (A_L\boldsymbol{u}_+ + A_R\boldsymbol{u}_-)\exp[ik_a z] + (R_L\boldsymbol{u}_- + R_R\boldsymbol{u}_+)\exp[-ik_a z], \quad z \leqq 0 \tag{8a}$$

$$\boldsymbol{H}(z) = (i/\eta_a)\,\{(-A_L\boldsymbol{u}_+ + A_R\boldsymbol{u}_-)\exp[ik_a z] + (-R_L\boldsymbol{u}_- + R_R\boldsymbol{u}_+)\exp[-ik_a z]\}, \quad z \leqq 0 \tag{8b}$$

where $k_a = +\,\omega\sqrt{(\mu_a\varepsilon_a)}$ is the wavenumber and $\eta_a = +\sqrt{(\mu_a/\varepsilon_a)}$ is the intrinsic medium impedance. In the classical optics manner, the subscripts L and R are indicative of left- and right-circular polarization, respectively. Thus,

$$\underline{X}^+(0) = \text{column}\,[(A_L + R_R), (-i/\eta_a)(A_L - R_R)], \tag{9a}$$

$$\underline{X}^-(0) = \text{column}\,[(A_R + R_L), (i/\eta_a)(A_R - R_L)]. \tag{9b}$$

In the region $z \geqq d$, the homogeneous achiral medium is specified by

$$\varepsilon(z) = \varepsilon_b, \quad \mu(z) = \mu_b, \quad \beta(z) = 0, \quad z \geqq d. \tag{10}$$

This region supports the transmitted plane wave given as

$$\boldsymbol{E}(z) = (T_L\boldsymbol{u}_+ + T_R\boldsymbol{u}_-)\exp[ik_b(z-d)], \quad z \geqq d \tag{11a}$$

$$\boldsymbol{H}(z) = (i/\eta_b)\,(-T_L\boldsymbol{u}_+ + T_R\boldsymbol{u}_-)\exp[ik_b(z-d)], \quad z \geqq d \tag{11b}$$

where $k_b = +\,\omega\sqrt{(\mu_b\varepsilon_b)}$ and $\eta_b = +\sqrt{(\mu_b/\varepsilon_b)}$. Hence,

$$\underline{X}^+(d) = \text{column}\,[T_L, -iT_L/\eta_b], \quad \underline{X}^-(d) = \text{column}\,[T_R, iT_R/\eta_b]. \tag{12a, b}$$

It follows from (6) that the effect of the chiral slab must be taken into account via constant matrices $\underline{\underline{S}}^\pm$, given as

$$\underline{\underline{S}}^\pm = \begin{pmatrix} S^{\pm,11} & S^{\pm,12} \\ S^{\pm,21} & S^{\pm,22} \end{pmatrix} \tag{13}$$

that will be decided upon shortly. With

$$\underline{X}^\pm(d) = \underline{\underline{S}}^\pm \cdot \underline{X}^\pm(0), \tag{14}$$

after using (9a, b) and (12a, b) it can be shown that

$$R_R = -\frac{(S^{+,11} - i\eta_a^{-1}S^{+,12}) - i\eta_b(S^{+,21} - i\eta_a^{-1}S^{+,22})}{(S^{+,11} + i\eta_a^{-1}S^{+,12}) - i\eta_b(S^{+,21} + i\eta_a^{-1}S^{+,22})} \cdot A_L, \tag{15a}$$

$$T_L = \frac{2\eta_b\eta_a^{-1}(S^{+,11}S^{+,22} - S^{+,12}S^{+,21})}{(S^{+,11} + i\eta_a^{-1}S^{+,12}) - i\eta_b(S^{+,21} + i\eta_a^{-1}S^{+,22})} \cdot A_L, \tag{15b}$$

$$R_L = -\frac{(S^{-,11} + i\eta_a^{-1}S^{-,12}) + i\eta_b(S^{-,21} + i\eta_a^{-1}S^{-,22})}{(S^{-,11} - i\eta_a^{-1}S^{-,12}) + i\eta_b(S^{-,21} - i\eta_a^{-1}S^{-,22})} \cdot A_R, \tag{15c}$$

$$T_R = \frac{2\eta_b\eta_a^{-1}(S^{-,11}S^{-,22} - S^{-,12}S^{-,21})}{(S^{-,11} - i\eta_a^{-1}S^{-,12}) + i\eta_b(S^{-,21} - i\eta_a^{-1}S^{-,22})} \cdot A_R. \tag{15d}$$

Once the matrices $\underline{\underline{S}}^\pm$ are determined, the reflection and transmission problem is solved.

4. System matrices $\underline{\underline{S}}^\pm$ for a linearly inhomogeneous chiral slab: First method

It was stated that the chiral slab has its constitutive properties varying as linear functions of z; thus,

$$\varepsilon(z) = \varepsilon_c(1 + z\Delta_\varepsilon), \quad 0 \leqq z \leqq d \tag{16a}$$

$$\mu(z) = \mu_c(1 + z\Delta_\mu), \quad 0 \leqq z \leqq d \tag{16b}$$

$$\beta(z) = \beta_c(1 + z\Delta_\beta), \quad 0 \leqq z \leqq d \tag{16c}$$

where Δ_ε, etc. have the unit of inverse length. Provided

$$d|\Delta_\varepsilon| \leqq 0.05, \quad d|\Delta_\mu| \leqq 0.05, \quad d|\Delta_\beta| \leqq 0.05, \quad |k_c\beta_c| < 0.1, \tag{17}$$

the matrix $\underline{\underline{\mathbf{B}}}(z)$ of (5) can be expressed, correct to order z, as

$$\underline{\underline{\mathbf{B}}}(z) = \underline{\underline{\mathbf{B}}}_0 + \underline{\underline{\mathbf{B}}}_1 z, \quad 0 \leqq z \leqq d \tag{18 a}$$

where

$$\underline{\underline{\mathbf{B}}}_0 = (1 - k_c^2 \beta_c^2)^{-1} \begin{pmatrix} i k_c^2 \beta_c & -\omega \mu_c \\ \omega \varepsilon_c & i k_c^2 \beta_c \end{pmatrix}, \tag{18 b}$$

and

$$\underline{\underline{\mathbf{B}}}_1 = (1 - k_c^2 \beta_c^2)^{-2} \begin{pmatrix} i k_c^2 \beta_c [\Delta_\mu + \Delta_\varepsilon + (1 + k_c^2 \beta_c^2) \Delta_\beta] & -\omega \mu_c [\Delta_\mu + k_c^2 \beta_c^2 (2 \Delta_\beta + \Delta_\varepsilon)] \\ \omega \varepsilon_c [\Delta_\varepsilon + k_c^2 \beta_c^2 (2 \Delta_\beta + \Delta_\mu)] & i k_c^2 \beta_c [\Delta_\mu + \Delta_\varepsilon + (1 + k_c^2 \beta_c^2) \Delta_\beta] \end{pmatrix} \tag{18 c}$$

are constant matrices with $k_c = +\omega \sqrt{(\mu_c \varepsilon_c)}$. The elements of $(\underline{\underline{\mathbf{B}}}_0 + \underline{\underline{\mathbf{B}}}_1 d)$ are within 1% of the corresponding elements of $\underline{\underline{\mathbf{B}}}(d)$ if equalities are used in the conditions (17).

The matrizants $\underline{\underline{\mathbf{M}}}^{\pm}(z)$, $0 \leqq z \leqq d$, may be recursively constructed as [19]

$$\underline{\underline{\mathbf{M}}}^{\pm}(z) = \sum_{m=0,1,2,\ldots,\infty} (\underline{\underline{\mathbf{M}}}^{\pm,m} z^m), \quad 0 \leqq z \leqq d \tag{19}$$

with

$$\underline{\underline{\mathbf{M}}}^{\pm,0} = \underline{\underline{\mathbf{I}}}, \quad \underline{\underline{\mathbf{M}}}^{\pm,1} = \pm \underline{\underline{\mathbf{B}}}_0, \tag{20 a, b}$$

$$\underline{\underline{\mathbf{M}}}^{\pm,m} = \pm (\underline{\underline{\mathbf{B}}}_1 \cdot \underline{\underline{\mathbf{M}}}^{\pm,m-2} + \underline{\underline{\mathbf{B}}}_0 \cdot \underline{\underline{\mathbf{M}}}^{\pm,m-1})/m, \quad m \geqq 2, \tag{20 c}$$

being constant matrices, and $\underline{\underline{\mathbf{I}}}$ is the identity matrix.

The solution of (6) in the region $0 \leqq z \leqq d$ can be obtained from (19) and (20 a–c) as

$$\begin{aligned} \underline{\underline{\mathbf{M}}}^{\pm}(z) \cong{} & \exp[\pm \underline{\underline{\mathbf{B}}}_0 z] \pm (1/2) \underline{\underline{\mathbf{B}}}_1 z^2 \\ & + (1/6) [2 \underline{\underline{\mathbf{B}}}_1 \cdot \underline{\underline{\mathbf{B}}}_0 + \underline{\underline{\mathbf{B}}}_0 \cdot \underline{\underline{\mathbf{B}}}_1] z^3 \\ & + \ldots \text{ higher order terms, } 0 \leqq z \leqq d. \end{aligned} \tag{21}$$

By diagonalizing [7, 19] $\underline{\underline{\mathbf{B}}}_0$ in (21) and using the fact that $\underline{\underline{\mathbf{S}}}^{\pm} = \underline{\underline{\mathbf{M}}}^{\pm}(d)$, it can be shown that

$$\begin{aligned} \underline{\underline{\mathbf{S}}}^{\pm} \cong{} & \underline{\underline{\mathbf{T}}}_c \cdot \exp[\pm i \underline{\underline{\mathbf{G}}}_c d] \cdot \underline{\underline{\mathbf{T}}}_c^{-1} \pm (1/2) \underline{\underline{\mathbf{B}}}_1 d^2 \\ & + (1/6) [2 \underline{\underline{\mathbf{B}}}_1 \cdot \underline{\underline{\mathbf{B}}}_0 + \underline{\underline{\mathbf{B}}}_0 \cdot \underline{\underline{\mathbf{B}}}_1] d^3 \\ & + \ldots \text{ higher order terms,} \end{aligned} \tag{22 a}$$

where

$$\underline{\underline{\mathbf{G}}}_c = \begin{pmatrix} k_c/(1 - k_c \beta_c) & 0 \\ 0 & -k_c/(1 + k_c \beta_c) \end{pmatrix}, \tag{22 b}$$

$$\underline{\underline{\mathbf{T}}}_c = \begin{pmatrix} 1 & 1 \\ -i \sqrt{(\varepsilon_c/\mu_c)} & i \sqrt{(\varepsilon_c/\mu_c)} \end{pmatrix}, \tag{22 c}$$

and $\underline{\underline{\mathbf{T}}}_c^{-1}$ is the inverse of $\underline{\underline{\mathbf{T}}}_c$.

If the matrices $\underline{\underline{\mathbf{B}}}_0$ and $\underline{\underline{\mathbf{B}}}_1$ of (18 b, c) commute, then (19) and (20 a–c) lead to an especially simple form for the matrizants $\underline{\underline{\mathbf{M}}}^{\pm}(z)$. This happens when $\Delta_\varepsilon = \Delta_\mu$, i.e., when the chiral slab has a constant value for $\mu(z)/\varepsilon(z)$ [20], $0 \leqq z \leqq d$. In this case, the recursive process (20 a, b) yields

$$\underline{\underline{\mathbf{M}}}^{\pm}(z) = \exp[\pm z(\underline{\underline{\mathbf{B}}}_0 + \underline{\underline{\mathbf{B}}}_1 z/2)]. \tag{23}$$

Diagonalization of the matrices $\pm (\underline{\underline{\mathbf{B}}}_0 z + \underline{\underline{\mathbf{B}}}_1 z^2/2)$ in this instance gives

$$\underline{\underline{\mathbf{M}}}^{\pm}(z) = \underline{\underline{\mathbf{T}}}^{\pm}(z) \cdot \exp[\underline{\underline{\mathbf{G}}}^{\pm}(z)] \cdot [\underline{\underline{\mathbf{T}}}^{\pm}(z)]^{-1}, \tag{24 a}$$

with

$$\underline{\underline{\mathbf{G}}}^{\pm}(z) = \begin{pmatrix} g_{1\pm}(z) & 0 \\ 0 & g_{2\pm}(z) \end{pmatrix}, \tag{24 b}$$

$$\underline{\underline{\mathbf{T}}}^{\pm}(z) = \begin{pmatrix} 1 & 1 \\ t_{1\pm}(z) & t_{2\pm}(z) \end{pmatrix}. \tag{24 c}$$

Here,

$$\begin{aligned} g_{1\pm}(z) + g_{2\pm}(z) = {} & \pm i k_c^2 \beta_c z [2(1 - k_c^2 \beta_c^2) + 2 z \Delta_\mu \\ & + z \Delta_\beta (1 + k_c^2 \beta_c^2)] \cdot [1 - k_c^2 \beta_c^2]^{-2}, \end{aligned} \tag{25 a}$$

$$\begin{aligned} 4 g_{1\pm}(z) \cdot g_{2\pm}(z) = {} & k_c^2 z^2 [4(1 - k_c^2 \beta_c^2) + (4 + z \Delta_\mu) z \Delta_\mu \\ & + k_c^2 \beta_c^2 (4 - z \Delta_\beta) z \Delta_\beta] \cdot [1 - k_c^2 \beta_c^2]^{-2}, \end{aligned} \tag{25 b}$$

$$t_{n\pm}(z) = \frac{\pm i k_c^2 \beta_c [2(1 - k_c^2 \beta_c^2) + 2 z \Delta_\mu + z \Delta_\beta (1 + k_c^2 \beta_c^2)] z - 2(1 - k_c^2 \beta_c^2)^2 g_{n\pm}(z)}{\pm \omega \mu_c [2(1 - k_c^2 \beta_c^2) + 2 z \Delta_\beta k_c^2 \beta_c^2 + z \Delta_\mu (1 + k_c^2 \beta_c^2)] z}, \quad n = 1, 2, \tag{25 c}$$

direct expressions for $g_{n\pm}(z)$, $n = 1, 2$, being too cumbersome to reproduce here. Again, for this special case,

$$\underline{\underline{\mathbf{S}}}^{\pm} = \underline{\underline{\mathbf{T}}}^{\pm}(d) \cdot \exp[\underline{\underline{\mathbf{G}}}^{\pm}(d)] \cdot [\underline{\underline{\mathbf{T}}}^{\pm}(d)]^{-1} \tag{26}$$

may be used in (15 a–d); further $S^{\pm,11} = S^{\pm,22}$ and $S^{\pm,12}/S^{\pm,21} = -\mu_c/\varepsilon_c$ in this special case.

5. System matrices $\underline{\underline{\mathbf{S}}}^{\pm}$ for a linearly inhomogeneous chiral slab: Second method

A second procedure to obtain $\underline{\underline{\mathbf{S}}}^{\pm}$, regardless of the commutative relationship of $\underline{\underline{\mathbf{B}}}_0$ and $\underline{\underline{\mathbf{B}}}_1$, follows from two observations. First, the numerators of the diagonal elements of $\underline{\underline{\mathbf{B}}}(z)$, $0 \leqq z \leqq d$, are directly proportional to $\beta(z)$, and would be identically zero were the slab to be achiral. Second, the diagonal elements are equal to each other. Together, these two observations suggest the transformation

$$\underline{X}^{\pm}(z) = \exp[f^{\pm}(z)]\,\underline{Y}^{\pm}(z), \quad 0 \leqq z \leqq d, \tag{27}$$

where the functions $f^{\pm}(z)$ are determined as follows. Substitution of (18a–c) and (27) in (5) leads to

$$(d/dz)\,\underline{Y}^{\pm}(z) = [-(d/dz)\,f^{\pm}(z)\,\underline{\underline{I}} \pm \underline{\underline{B}}_0 \pm z\,\underline{\underline{B}}_1] \cdot \underline{Y}^{\pm}(z), \quad 0 \leqq z \leqq d. \tag{28}$$

The aim now is to make the diagonal elements of the composite matrix on the right-hand side of (28) equal to zero; hence,

$$f^{\pm}(z) = \pm\, i(k_c^2 \beta_c z/2)\,[2(1-k_c^2\beta_c^2) + z(\Delta_\mu + \Delta_\varepsilon + \Delta_\beta) + z\,\Delta_\beta k_c^2 \beta_c^2] \cdot [1-k_c^2\beta_c^2]^{-2}, \tag{29}$$

and

$$(d/dz)\,\underline{Y}^{\pm}(z) = \pm\,[\underline{\underline{C}}_0 + z\,\underline{\underline{C}}_1] \cdot \underline{Y}^{\pm}(z), \; 0 \leqq z \leqq d, \tag{30a}$$

with

$$\underline{\underline{C}}_0 = (1-k_c^2\beta_c^2)^{-1}\begin{pmatrix} 0 & -\omega\mu_c \\ \omega\varepsilon_c & 0 \end{pmatrix}, \tag{30b}$$

and

$$\underline{\underline{C}}_1 = (1-k_c^2\beta_c^2)^{-2} \begin{pmatrix} 0 & -\omega\mu_c[\Delta_\mu + k_c^2\beta_c^2(2\Delta_\beta + \Delta_\varepsilon)] \\ \omega\varepsilon_c[\Delta_\varepsilon + k_c^2\beta_c^2(2\Delta_\beta + \Delta_\mu)] & 0 \end{pmatrix}. \tag{30c}$$

It is to be noted that $\underline{\underline{C}}_0$ and $\underline{\underline{C}}_1$ are traceless matrices.

The matrizants $\underline{\underline{N}}^{\pm}(z)$ of (30a) satisfy the differential equations $(d/dz)\,\underline{\underline{N}}^{\pm}(z) = \pm\,[\underline{\underline{C}}_0 + \underline{\underline{C}}_1 z] \cdot \underline{\underline{N}}^{\pm}(z)$, $0 \leqq z \leqq d$, and can be obtained from the same procedure as for $\underline{\underline{M}}^{\pm}(z)$ in the previous section; thus,

$$\underline{\underline{N}}^{\pm}(z) \cong \exp[\pm\,\underline{\underline{C}}_0 z] \pm (1/2)\,\underline{\underline{C}}_1 z^2 + (1/6)\,[2\,\underline{\underline{C}}_1 \cdot \underline{\underline{C}}_0 + \underline{\underline{C}}_0 \cdot \underline{\underline{C}}_1]\,z^3 + \ldots \text{ higher order terms}, \quad 0 \leqq z \leqq d, \tag{31}$$

whence

$$\underline{\underline{S}}^{\pm} \cong \exp[f^{\pm}(d)]\,\underline{\underline{N}}^{\pm}(d), \tag{32}$$

after noting that $\underline{Y}^{\pm}(0) = \underline{X}^{\pm}(0)$ and making use of (14) and (27). Upon diagonalizing $\underline{\underline{C}}_0$ in (31), it can be shown that

$$\underline{\underline{S}}^{\pm} \cong \exp[f^{\pm}(d)]\,\{\underline{\underline{T}}_c \cdot \exp[\pm\, i\,\underline{\underline{K}}_c k_c d/(1-k_c^2\beta_c^2)] \cdot \underline{\underline{T}}_c^{-1} \pm (1/2)\,\underline{\underline{C}}_1 d^2 + (1/6)\,[2\,\underline{\underline{C}}_1 \cdot \underline{\underline{C}}_0 + \underline{\underline{C}}_0 \cdot \underline{\underline{C}}_1]\,d^3 + \ldots \text{ higher order terms}\}, \tag{33a}$$

where

$$\underline{\underline{K}}_c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{33b}$$

When $\Delta_\varepsilon = \Delta_\mu$, then $\underline{\underline{C}}_0$ and $\underline{\underline{C}}_1$ commute with each other. The matrizants $\underline{\underline{N}}^{\pm}(z)$ for this special case can be obtained exactly as

$$\underline{\underline{N}}^{\pm}(z) = \exp[\pm\, z(\underline{\underline{C}}_0 + \underline{\underline{C}}_1 z/2)], \tag{34}$$

and after diagonalization of $\pm\, z(\underline{\underline{C}}_0 + \underline{\underline{C}}_1 z/2)$ as

$$\underline{\underline{N}}^{\pm}(z) = \underline{\underline{T}}_c \cdot \exp[\pm\, i h(z)\,\underline{\underline{K}}_c] \cdot \underline{\underline{T}}_c^{-1}, \tag{35a}$$

where

$$h(z) = (k_c z/2)\,[2(1-k_c^2\beta_c^2) + z\,\Delta_\mu(1+k_c^2\beta_c^2) + 2z\,\Delta_\beta k_c^2\beta_c^2] \cdot [1-k_c^2\beta_c^2]^{-2}. \tag{35b}$$

Consequently, for this special case

$$\underline{\underline{S}}^{\pm} = \exp[\pm\, b]\,\underline{\underline{T}}_c \cdot \exp[\pm\, i h(d)\,\underline{\underline{K}}_c] \cdot \underline{\underline{T}}_c^{-1}, \tag{36a}$$

with

$$b = i(k_c^2\beta_c d/2)\,[2(1-k_c^2\beta_c^2) + d(2\Delta_\mu + \Delta_\beta) + d\Delta_\beta k_c^2\beta_c^2] \cdot [1-k_c^2\beta_c^2]^{-2}. \tag{36b}$$

Conclusions

Comparison of the matrices $\underline{\underline{B}}_0$ and $\underline{\underline{C}}_0$, as well as of $\underline{\underline{B}}_1$ and $\underline{\underline{C}}_1$, shows that the matrices $\underline{\underline{S}}^{\pm}$ of (33) and (36) may be computationally preferable than the $\underline{\underline{S}}^{\pm}$ of (22) and (26), although the difference in the computational efforts for the two schemes is only slight for the 2×2 matrix equations involved here. Further, it should be noted that the right sides of (22a) and (33a) may converge very slowly for electrically thick inhomogeneous slabs.

As an example of the methods developed above, shown in figs. 1–4 are computations of R_L/A_R, R_R/A_L, T_R/A_R, and T_L/A_L performed (on a Macintosh II desktop computer using Absoft MacFortran/020) when $\varepsilon_a = \varepsilon_b = \varepsilon_0$, $\mu_a = \mu_b = \mu_c = \mu_0$, $\varepsilon_c = 3\,\varepsilon_0$ and $\beta_c/d = 5\times 10^{-3}$ as functions of the normalized frequency $k_a d$. The solid lines are for an inhomogeneous slab with $\Delta_\varepsilon/d = \Delta_\mu/d = 0.05$ and $\Delta_\beta/d = 0.01$, while the dashed lines are for the corresponding homogeneous slab ($\Delta_\varepsilon/d = \Delta_\mu/d = \Delta_\beta/d = 0.0$). Since the planewave incidence is normal and all three media are lossless, adequacy of the computed solutions was checked by ascertaining the satisfaction of the energy conservation conditions,

$$|R_L/A_R|^2 + (\eta_a/\eta_b)\,|T_R/A_R|^2 = 1, \tag{37a}$$

$$|R_R/A_L|^2 + (\eta_a/\eta_b)\,|T_L/A_L|^2 = 1, \tag{37b}$$

within a $\pm\,0.1\,\%$ error.

It should be noted that for the computational parameters of figs. 1–4, the matrices $\underline{\underline{B}}_0$ and $\underline{\underline{B}}_1$ commute, and so do the matrices $\underline{\underline{C}}_0$ and $\underline{\underline{C}}_1$. Use of (22), (26), (33) or (36) yielded identical and correct results for homogeneous slabs at all appropriate frequencies. But the use of (22a) and (33a) with terms upto order d^3 was found satisfactory only for the frequency range $0 \leqq k_a d \leqq 1.0$ for the inhomogeneous slab. On the other hand, using (26) or (36) yielded adequate and identical results for the inhomogeneous slab. It is observed that the reflected power densities $|R_L/A_R|^2 = |R_R/A_L|^2$ and the transmitted power densities satisfy the condition $|T_L/A_L|^2 = |T_R/A_R|^2$ as expected since $\mu(z)/\varepsilon(z)$, $0 \leq z \leq d$, is constant and the plane waves are normally incident [20, 21], whether the chiral slab is inhomogeneous or not. The phases of T_L/A_L and T_R/A_R are different from each other, but both R_L/A_R and R_R/A_L have the same phases, whether or not the chiral slab is inhomogeneous. Finally, from these four figures it

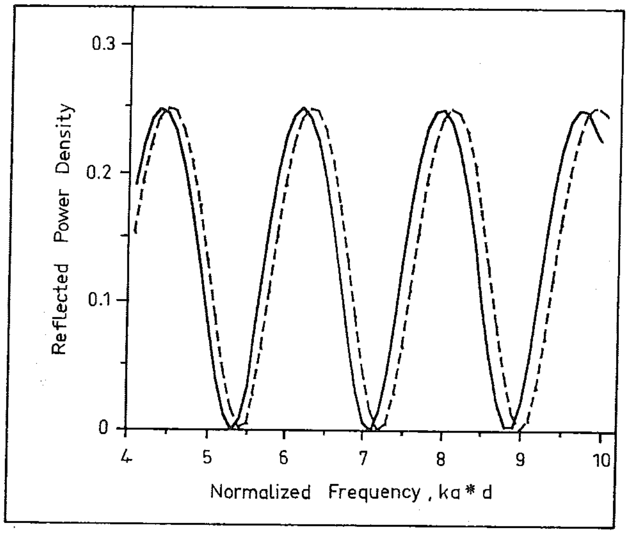

Fig. 1. Normalized reflected power density $|R_L/A_R|^2 = |R_R/A_L|^2$ as a function of the normalized frequency $k_a d$ when $\varepsilon_a = \varepsilon_b = \varepsilon_0$, $\mu_a = \mu_b = \mu_c = \mu_0$, $\varepsilon_c = 3\varepsilon_0$, $\beta_c/d = 5 \times 10^{-3}$. Solid line (—) $\Delta_\varepsilon/d = \Delta_\mu/d = 0.05$ and $\Delta_\beta/d = 0.01$. Dashed line (---) $\Delta_\varepsilon/d = \Delta_\mu/d = \Delta_\beta/d = 0.0$.

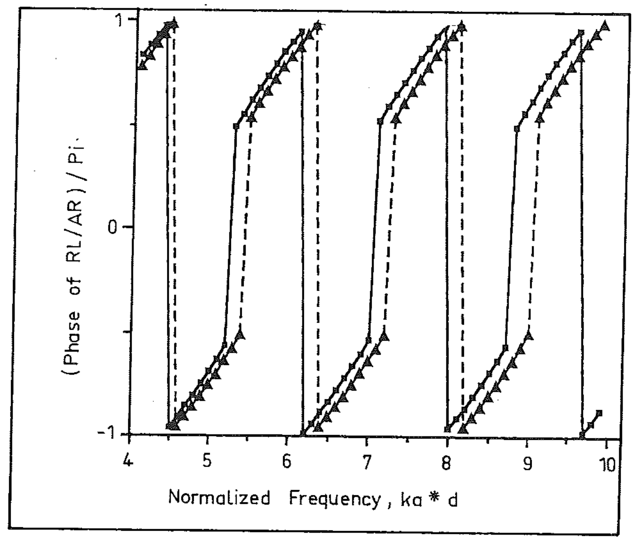

Fig. 2. Phase of R_L/A_R $(= R_R/A_L)$ normalized by π as a function of the normalized frequency $k_a d$ when $\varepsilon_a = \varepsilon_b = \varepsilon_0$, $\mu_a = \mu_b = \mu_c = \mu_0$, $\varepsilon_c = 3\varepsilon_0$, $\beta_c/d = 5 \times 10^{-3}$. Solid line (—) $\Delta_\varepsilon/d = \Delta_\mu/d = 0.05$ and $\Delta_\beta/d = 0.01$. Dashed line (---) $\Delta_\varepsilon/d = \Delta_\mu/d = \Delta_\beta/d = 0.0$.

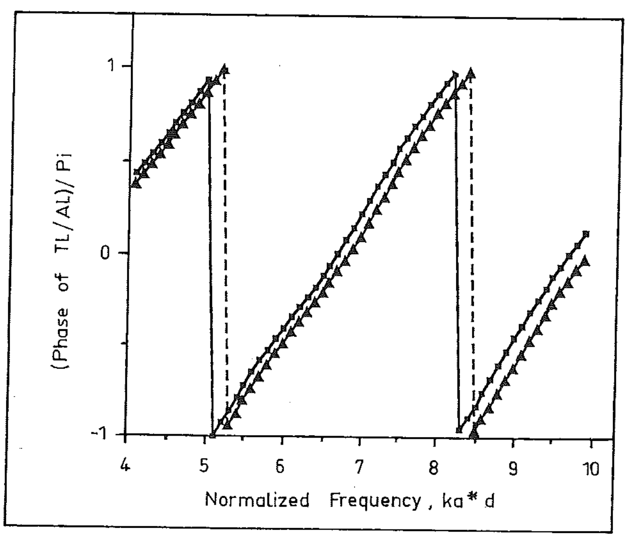

Fig. 3. Phase of T_L/A_L normalized by π as a function of the normalized frequency $k_a d$ when $\varepsilon_a = \varepsilon_b = \varepsilon_0$, $\mu_a = \mu_b = \mu_c = \mu_0$, $\varepsilon_c = 3\varepsilon_0$, $\beta_c/d = 5 \times 10^{-3}$. Solid line (—) $\Delta_\varepsilon/d = \Delta_\mu/d = 0.05$ and $\Delta_\beta/d = 0.01$. Dashed line (---) $\Delta_\varepsilon/d = \Delta_\mu/d = \Delta_\beta/d = 0.0$.

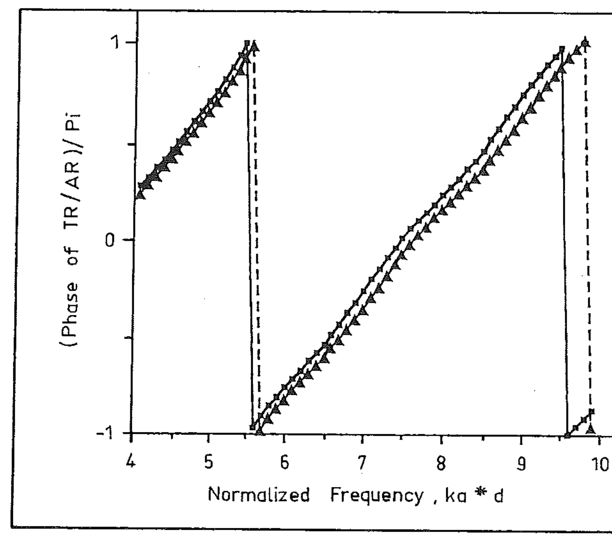

Fig. 4. Phase of T_R/A_R normalized by π as a function of the normalized frequency $k_a d$ when $\varepsilon_a = \varepsilon_b = \varepsilon_0$, $\mu_a = \mu_b = \mu_c = \mu_0$, $\varepsilon_c = 3\varepsilon_0$, $\beta_c/d = 5 \times 10^{-3}$. Solid line (—) $\Delta_\varepsilon/d = \Delta_\mu/d = 0.05$ and $\Delta_\beta/d = 0.01$. Dashed line (---) $\Delta_\varepsilon/d = \Delta_\mu/d = \Delta_\beta/d = 0.0$.

is clear that the influence of inhomogeneity on the plane wave reflection and transmission characteristics increases as the chiral slab becomes electrically thicker.

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