

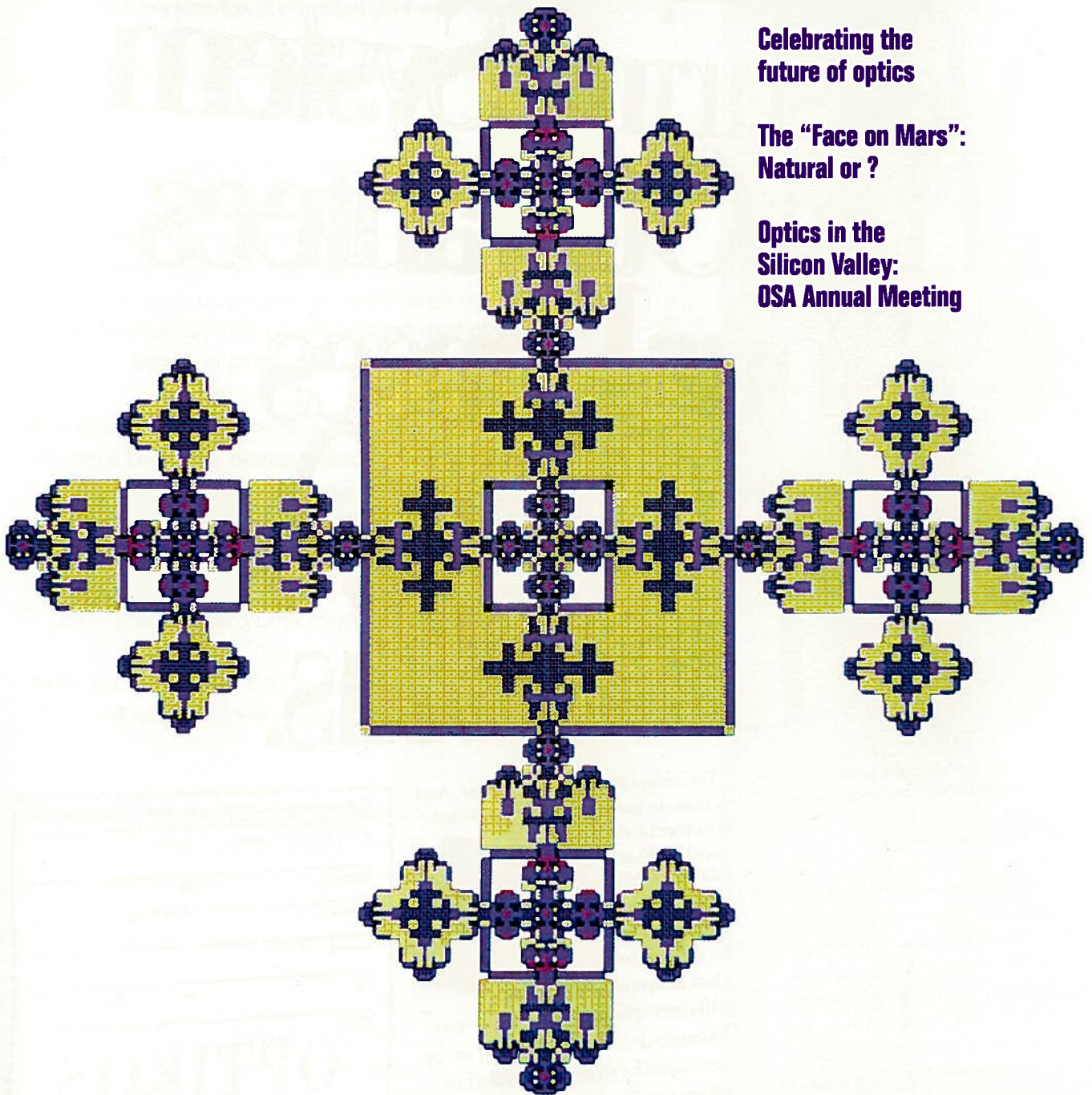
OPTICS & PHOTONICS

N E W S
Published by the Optical Society of America June 1991 Vol. 2, No. 6

**Celebrating the
future of optics**

**The "Face on Mars":
Natural or ?**

**Optics in the
Silicon Valley:
OSA Annual Meeting**



Fractals, continued fractions, and sequences

Unsuspected dangers truncated

*"Most dangerous
Is that temptation that goads us on
To sin in loving virtue."*

William Shakespeare,
Measure for Measure
Act II, Scene II

Limits and differentiability

Differential calculus is central to the physical sciences. Analyses of phenomena described by Maxwell's equations, Navier's equation, the Navier-Stokes equations, or the Korteweg-deVries equation, etc., all need differential calculus. Limits are essential to differential calculus. Differentiation of a function $f(x)$ about a point x_0 , as a casual perusal of calculus textbooks^{1,2} as well as Newton's notebooks will suffice to show,³ has a geometric interpretation in which the spread $(x_b - x_a)$ of the range $x_a \leq x_0 \leq x_b$ is made increasingly small until it *almost vanishes*. Assuming that $f(x)$ is continuous, this limiting process yields its derivative at x_0 . Or, does it?

A good example for opticspeople⁴ is furnished by the reflection and transmission of light at a rough bimaterial interface $S(x,y,z) = 0$. The mathematical expression of Huygens's principle⁵⁻⁷ requires that the unit normal to the interface be known everywhere on the interface. This unit normal can be obtained as $\nabla S/|\nabla S|$, and thus requires the partial derivatives of $S(x,y,z)$. Assuming that $S(x,y,z)$ is a smooth continuous function over the dependent variable ranges considered, can the normal to the interface everywhere be *unambiguously* found? Can it be found, in some

instances,
at all?

There is no particular reason to resurrect the Newton versus Leibniz controversy here in its gory details. It is sufficient to mention that the Hanoverian Royal Genealogist was troubled by the limiting procedure, but the views of the English Master of the Mint carried the day. Indeed, Newtonians held firmly to their ground until the middle of the last century and have begun to yield only quite recently.

Scattering by a crenellated fiber

Scattering by infinitely long fibers is another example of interest to opticspeople. A cylinder whose cross-section is bounded by a smooth contour can be handled using a variety of techniques.^{8,9} Consider, however, a simply-connected cross-section for which the method of constructing

FRACTALS, CONTINUED
AND SEQUENCES
FRACTALS, CONTINUED

of extrapolating from analyses

FRACTIONS, AND SEQUENCES
FRACTALS, CONTINUED

By Akhlesh Lakhtakia and Richard S. Andrusis Jr.

This figure is entitled "G² MOD3 A(1,1)=0" and is a variant of the snowflake shown on the cover. If the A(1,1) term of the matrix used to generate the elemental figure is set equal to zero before the matrix is formed, Snowflake II results. D.E. Passoja⁹. All rights reserved 1991, New York, N.Y. and Penn State University.

the bounding contour takes after the Koch snowflake^{10,11} in the following fashion.¹²

To begin, each side of an equilateral triangle is partitioned into (2k+1) segments of equal size, $k \geq 1$, consecutively numbered 1,2,3,..., 2k+1. Then, on the even-numbered segments of each side are constructed equilateral triangles whose interiors lie external to the region enclosed by the base triangle. Finally, the even-numbered line segments are deleted from the original triangle, thereby yielding a polygon with 3(3k+1) sides. This basic construction is then repeated on all of the sides of this new polygon, and so on ad nauseam. At the stage $n = 0$, only the initial triangle is there; at the stage $n = 1$, the bounding contour is a 3(3k+1)-polygon, and so on. At the n-th stage, the cross-sectional area A_n is related to the area A_0 of the

initial triangle by the relation

$$A_n = A_0 \{ 1 + 3k [3k+1]^{-1} \sum_{m \in \{1,2,\dots,n\}} [3k+1]^m [2k+1]^{-2m} \}, \quad (1)$$

which, in the limit $n \rightarrow \infty$, goes to $A_\infty = A_0 4(k+1)[4k+1]^{-1}$, and is clearly finite. Further, it should be noted that $A_\infty \leq (8/5) A_0$ for all k, and the contour at the n-th stage is continuous everywhere.

Application of Huygens's principle requires that certain integrations be performed over the contour. It also requires that the unit outward normal to the bounding contour be known everywhere. The perimeter P_n of the contour at the n-th stage is related to the perimeter P_0 of the initial triangle by the relation

$$P_n = \{ [3k+1]/[2k+1] \}^n P_0, \quad (2)$$

from where it is obvious that for a given (2k+1)-equisecting, the perimeter grows unboundedly with n. Since the contour is extremely crenellated at any large enough n, one possible solution is to round off its vertices ever so slightly, thereby obtaining the unit normal unam-



Karl Theodor Wilhelm Weierstrass

biguously at all points. The contour integration would tax the resources of the most powerful computer available, but one could emulate Scarlett O'Hara and do with today's computers. However, will the calculations for a finite n , howsoever large, be the Holy Grail, *i.e.*, acceptably close to those for $n = \infty$?

The plane wave extinction cross-section of a non-absorbing cylinder is proportional to the square of the cross sectional area, provided the maximum linear extent of the cross sectional geometry is appreciably smaller than the wavelengths inside as well as outside the cylinder.^{13,14} Thus, an upper bound on the extinction cross section for $n = \infty$ is obtainable in the long-wavelength limit. But the scattering pattern depends on the shape of the bounding contour;^{13,14} hence, it may not be predictable from an analysis for finite n .

Infinite-stepped analyses

Many other areas of physical sciences and engineering require an infinite number of steps for analytical purposes. For example, (a) infinite series representations of Bessel and Hankel functions used in scattering theory,¹³⁻¹⁵ (b) infinite-sized continued fractions in electrode physics,¹⁶ and (c) infinite number of steps used for the evolution of the state of a physical system in stability analysis.^{17,18}

It is rarely possible to implement untruncated analyses for purposes of calculation. Further, the finite precision as well as the finite memory size of even the largest computer imaginable will end up introducing errors.

The aim, therefore, always is to be able to represent an infinite-stepped analysis as the limit of a finite-stepped analysis. Thus, the sequence of an increasingly longer finite-stepped analysis must be such as to uniformly converge to the infinite-stepped analysis, thereby reducing the errors-of-truncation to within acceptable tolerances. However, as this article will illustrate with three examples, the conclusions obtained from truncated analyses *may* not be correctly extrapolated onto those putatively from the untruncated analyses.

Example 1: Generalized series representations

Series representations are ubiquitous in almost all areas of mathematical physics. In the most generalized sense, it is possible to write a function $f(x)$ over some range, $x \in [x_1, x_2]$, as¹⁹

$$f(x) = \sum_{n \in \{1, \infty\}} \psi_n(x), \quad (3)$$

where the $\psi_n(x)$ are the basis functions. Let it be assumed that each $\psi_n(x)$ is continuous and continuously differentiable for $x \in [x_1, x_2]$, and the finite sum

$$f(x;N) = \sum_{n \in \{1, N\}} \psi_n(x), \quad (4)$$

be defined. It is also assumed now that the infinite series on the right side of Eq. (3) is uniformly convergent so that^{2,20}

$$\lim_{N \rightarrow \infty} f(x;N) = f(x), \quad (5)$$

Since the right side of Eq. (4) has a finite number of terms, it can be shown that $f(x;N)$ is differentiable term-by-term.² The problem really comes when it is asked if

$$\lim_{N \rightarrow \infty} (d/dx)f(x;N) = (d/dx)f(x), \quad (6)$$

for all x , for no x , or for some $x \in [x_1, x_2]$? In other words, will a term-by-term differentiation of the right side of Eq. (3) always/never/sometimes yield the differential of the left side of Eq. (3)?

The question has an interesting history. Ampère²¹ believed that continuity implied differentiability, and so did Cauchy,²² whose influence on the physical sciences is unmistakably strong. Ampère's defective proof went unchallenged for some 50 years, apart from an alleged 1861 assertion of Riemann that the sum

$$\sum_{n \in \{1, \infty\}} n^2 \sin(n^2 \pi x), \quad (7)$$

is continuous but non-differentiable for all x ; Riemann's assertion is only partially correct.^{20,23-25}

It should be noted that Eq. (7) is a lacunar Fourier series. With this in mind, duBois-Reymond attempted the construction of continuous Fourier series that are convergent everywhere, but are non-differentiable for all x .²⁶ Success, however, had already come to Weierstrass,²⁷ who showed in 1872 that the function

$$f_w(x) = \sum_{n \in \{1, \infty\}} a^n \cos(b^n \pi x), \quad (8)$$

is a *continuous but nowhere-differentiable* function provided b is an odd integer, $0 < a < 1$ and $ab > (2+3\pi)/2$. DuBois-Reymond, who first published Eq. (8),²⁸ described Weierstrass' discovery as too strange for both immediate



Helge Von Koch

perception and critical understanding (see also Ref. 29). An early history of the progeny of the Weierstrass function is available in Singh,²⁰ as also is the remarkable Singh family of continuous but non-differentiable functions.

Apart from the excitement $f_w(x)$ and similar functions caused in mathematical circles, physical sciences appear to have hardly taken note of this discovery. Differentiability of continuous functions has had almost the stature of a postulate in all branches of sciences where differential calculus is used. This situation began to change somewhat during the 1970s, when the highly expository work of Mandelbrot gave rise to what is now known as *fractal geometry*.³⁰ With particular emphasis on electromagnetic field theory, fractal investigations of propagation in turbulent media (e.g., Ref. 31), as well as their use in the imaging sciences,³² are nowadays being increasingly carried out.

Since $f_w(x)$ is generally intractable for physical sciences, it is not surprising that the function actually used is the truncation

$$f_w(x;N) = \sum_{n \in \{1,N\}} a^n \cos(b^n \pi x), \quad (9)$$

that can be differentiated term-by-term. Other continuous but non-differentiable functions may also be considered,²⁰ but only their truncated forms are differentiable everywhere. Of course, $f_w(x;N)$ is not a fractal function, but it is computationally tractable. The merits of replacing $f_w(x)$ by $f_w(x;N)$ with as large N as possible and still labeling the resulting analysis as fractal are debatable,³³⁻³⁵ while larger questions of the relevance of fractals to physics³⁶ and mathematics³⁷ are still being thrashed out. Nevertheless, it is necessary to appreciate that the differentiability of $f_w(x;N)$ does not hold when the limit $N \rightarrow \infty$ is taken, and that this feature will not vanish, regardless of the capabilities of the computer being used.

Example 2: An infinite ladder circuit

The second example in the present connection is more insidious. The representation of irrational numbers in the Stieltjes continued fraction form inspired the development of ladder circuits for electric networks.³⁸ Continued fractions have recently been used for understanding fractal quantization of particles in one-dimensional potentials with incommensurate periods,³⁹ as well as in two-dimensional electron gases⁴⁰ (see also Refs. 41, 42). Examination of the

frustrated instabilities of active optical resonators⁴³ and characterization of rough surfaces^{18,44,45} are other recent uses of continued fractions.

Consider an L-C ladder circuit, whose input impedance $Z^{(i)}(\omega)$ is specified via the recursion rule

$$Z^{(i+1)}(\omega) = j\omega L + 1/[j\omega C + 1/Z^{(i)}(\omega)]; i > 1, \quad (10a)$$

$$Z^{(0)}(\omega) = j\omega L + 1/[j\omega C], \quad (10b)$$

where the positive ω is the circular frequency and $j = \sqrt{-1}$. Computation of $Z^{(N)}(\omega)$, for arbitrarily large values of N , can be performed using the theory of continued fractions.^{45,46}

Let us consider the resonance frequencies of $Z^{(N)}(\omega)$; at a resonance frequency, the imaginary part of $Z^{(N)}(\omega)$ is identically zero. It can be shown by direct computation that the number of such resonance frequencies is equal to N for $N \neq \infty$. Consider, however, $Z^{(\infty)}(\omega)$ itself: it is a solution of the quadratic equation

$$Z^{(\infty)}(\omega) = j\omega L + 1/[j\omega C + 1/Z^{(\infty)}(\omega)]. \quad (11)$$

From Eq. (11), it can be seen that the imaginary part of $Z^{(\infty)}(\omega)$ is proportional to ω for $\omega^2 LC \leq 4$, and $Z^{(\infty)}(\omega)$ is purely imaginary for $\omega^2 LC > 4$. Given that $Z^{(\infty)}(\omega)$ begins with an inductance, it follows that $Z^{(\infty)}(\omega)$ has one and only one resonance: trivially, at $\omega = 0$.

Further, it can be observed from computations that all resonance frequencies of $Z^{(N)}(\omega)$ for $N \neq \infty$ are less than $2/\sqrt{LC}$ in magnitude. This observation is central to the present discussion. The impedance $Z^{(\infty)}(\omega)$ can be written as the infinite-sized continued fraction

$$Z^{(\infty)}(\omega) = j\omega L + \frac{1}{j\omega C + \frac{1}{j\omega L + \frac{1}{j\omega C + \frac{1}{j\omega L + \dots}}}} \quad (12)$$

A casual comparison of Eqs. (10a) and (12) makes it easy to believe that

$$Z^{(\infty)}(\omega) = \lim_{N \rightarrow \infty} Z^{(N)}(\omega). \quad (13)$$

But, according to the theory of continued fractions^{45,46}, $Z^{(N)}(\omega)$ will converge to $Z^{(\infty)}(\omega)$ as N increases, if and only if the twin conditions

$$|\omega C| \geq 2, \quad |\omega L| \geq 2 \quad (14)$$

are simultaneously satisfied. This means that Eq. (13) is an identity for $\omega^2 LC \geq 4$, and a simple comparison of Eqs. (10a) and (12), while fraught with practicality, is dangerous.

For $\omega^2 LC < 4$, the identity Eq. (13) is untrue,⁴⁷ but all resonance frequencies of $Z^{(\omega)}(\omega)$ and $Z^{(0)}(\omega)$ lie in this ω -regime. Hence, the resonance behaviour of $Z^{(\omega)}(\omega)$ cannot be predicted from that of $Z^{(0)}(\omega)$; nor can the number of resonance frequencies of $Z^{(\omega)}(\omega)$ be extrapolated from that of $Z^{(0)}(\omega)$.

Example 3: An infinite sequence

A connection of ladder circuits with the well-known Fibonacci sequence having been recently made,⁴⁸ the third example comes from number theory and may be of interest to cryptologists and communication theorists.⁴⁹ This example is also readily accessible using the gizmo called, nowadays somewhat euphemistically, the hand-held *calculator*.

Let us construct a sequence $\{G_n\}$, $n = 1, 2, 3, \dots$, of positive numbers defined as⁵⁰

$$G_{2m} = 2^{m-1}C_m - 2^{2m-1}C_{m-3} + 2^m C_{m-5}, \quad m = 1, 2, 3, \dots, \quad (15a)$$

$$G_{2m+1} = 2^m C_m - 2^{2m+1}C_{m-2} + 2^{2m} C_{m-5}, \quad m = 0, 1, 2, 3, \dots, \quad (15b)$$

where ${}^n C_k = n!/[k!(n-k)!]$; ${}^n C_k = 0$ for $k < 0$. This yields

$$\{G_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots\} \quad (15c)$$

making $\{G_n\}$ look like the standard Fibonacci sequence $\{F_n\}$, $n = 1, 2, 3, \dots$, given as

$$F_1 = 1; F_2 = 1; F_n = F_{n-1} + F_{n-2}, \quad n > 2 \quad (16)$$

Is $\{G_n\}$ congruent with $\{F_n\}$? One could tabulate the two sequences to answer this question, but that may not be a satisfactory enough procedure.⁵¹ Instead, a neater way⁵² is to look at the asymptotic values of G_n and F_n . Thus, while

$$\lim_{n \rightarrow \infty} (G_n/G_{n-1}) \sim 2, \quad (17)$$

the limit

$$\lim_{n \rightarrow \infty} (F_n/F_{n-1}) \sim (1 + \sqrt{5})/2, \quad (18)$$

clearly implies that for some N , $F_n < G_n$ for all $n > N$. Direct computation⁵¹ yields $N = 14$, but it is to be doubted, given Eq. (15c), that many people would have looked that far!

Summing up

For uniformly converging processes, it is possible that conclusions derived from a truncated analysis *may* not hold

as the putative conclusions of the untruncated analysis. This has been exemplified here by considering the differentiability of the Weierstrass function. It is also suggested that even undergraduate students be taught the difference between continuity and differentiability, because almost-everywhere-true notions acquired at such early stages are hard to get rid of at a later stage in a person's career.

The second example—that of the infinite L-C ladder circuit—is illustrative of another danger: That which appears to be converging may not be so. Finally, the third example shows that what appears to be true for a finite number of steps may not be asymptotically correct. Truly insidious!

Acknowledgements

The apparent inability of the Count of Sesame Street to count beyond a hundred acknowledged as the motivation for this work. The foregoing is, in a manner, a requiem for Baron Augustin Cauchy, that giant among scientists, who labored tirelessly to advance mathematical physics, but apparently never questioned Ampère's dictum: continuity implies differentiability.

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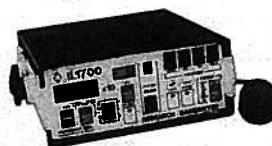
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Akhlesh Lakhtakia is an assistant professor and **Richard S. Andrulis Jr.** was an undergraduate student with the Department of Engineering Science and Mechanics, The Pennsylvania State University, University Park, Pa. Andrulis is now with the Department of Electrical Engineering at Cornell University, Ithaca, N.Y.

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8-D—Ph.D. (1991) in laser optics. Seeking research position subsequent to graduation in Oct. 1991. Hands-on experience with nonlinear fiber optics and erbium-doped fiber amplifiers. UK citizen.

8-E—M.S. (1987) in electrophysics. Seven years R&D experience in fiber optics, semiconductor laser fab, LiNbO₃ device fab, project management, laser system development. Patents, publications. Interested in nonlinear optics, OEICs, diode laser Silicon Valley/mid-Peninsula.

8-F—M.S. (1990) in optical engineering. Experience in fiber optic manufacturing and testing, solid-state devices, lasers, optical materials and components. Seeking engineering or R&D position.

8-G—Ph.D. (1989) in electrical engineering. Research experience in 3-D fluorescent microscopic imaging, digital image processing, CCD imaging, and optical information processing. Seeking R&D position in an industry or research lab.

8-H—B.S. (1991) in manufacturing engineering. Seeking a position as a sales, applications, or manufacturing engineer in the optics or laser industry. Have concentration in optics and lasers. Senior project involves opto-fluidic interface for Boeing's "Fly-By-Light" program. U.S. citizen. Willing to relocate.

8-I—Ph.D. (1991) in optics; M.Sc (1982) in electrical engineering. Over five years R&D experience in polarization-maintaining fibers, passive fiber optic devices, fiber optic measuring and sensing systems, both theoretical and experimental. Seeking R&D or academic position. Canadian landed immigrant.

✂

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Corrections

Fractals, continued fractions, and gremlins

Some gremlins crept into two references at the end of the June 1991 article by Akhlesh Lakhtakia and Richard S. Andrusis Jr. ("Fractals, continued fractions, and sequences," pages 8-13) and changed the mathematical symbols back to *OPN's* technical coding. The corrected references read as follows:

47. That $Z^{(*)}(\omega) \neq \text{Lim}_{N \rightarrow \infty} Z^{(N)}(\omega)$ everywhere in the circular region $\omega^2 LC < 4$ in the $\omega L - \omega C$ plane is not entirely correct. Truly, $Z^{(*)}(\omega) \neq \text{Lim}_{N \rightarrow \infty} Z^{(N)}(\omega)$ inside the square region bounded by the straight lines $\omega L = \pm 2$ and $\omega C = \pm 2$.

51. In the present instance, the incongruity of $\{G_n\}$ with $\{F_n\}$ can be established by tabulation rather easily. From direct computation, $G_n = F_n$ for $n \leq 14$, $G_{15} = 611 = 1 + F_{15}$ and $G_{16} = 989 = 2 + F_{16}$.

Getting the bugs out: Part II

In the process of getting the bugs out of our new desktop publishing operation, we lost a few bugs in a recent article on diamond technology (March *OPN*, page 48). Then, to compound the error, we printed an incomplete correction! The last paragraph on page 48, describing work underway at China Lake Naval Weapons Center, should have read: "At China Lake Naval Weapons Center, researchers are testing the use of diamond as a dome covering for infrared missiles. Usual infrared materials such as zinc sulfide and zinc selenide are friable and pit on impact with *dust, dirt, rain, and insects*. Other applications include diamond windows and aiming devices for x-rays, microwaves, and excimer and UV lasers."