

ON PATHOLOGICAL CONDITIONS AND FRESNEL COEFFICIENTS

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Pathological conditions at bimaterial interfaces have been examined by looking at the Fresnel reflection and transmission coefficients of optically smooth interfaces.

Reflection and refraction of plane waves at optically smooth interfaces of two dissimilar media have been widely studied for several centuries. Yet this simple boundary value problem continues to be of great interest to this day because of its extensive applicability. The aim of this communication is to study the pathological conditions of reflection and refraction at bimaterial interfaces.

Consider the optically smooth interface $z = 0$: The permittivity and the permeability of the medium occupying the zone $z \leq 0$ are denoted by the positive real scalars ϵ_0 and μ_0 , respectively; those of the medium in the zone $z \geq 0$ are denoted by the positive real scalars ϵ and μ . The wavenumbers, $k_0 = \omega\sqrt{(\epsilon_0 \mu_0)}$ and $k = \omega\sqrt{(\epsilon \mu)}$, and the intrinsic impedances, $\eta_0 = \sqrt{(\mu_0/\epsilon_0)}$ and $\eta = \sqrt{(\mu/\epsilon)}$, are defined for the two media; an $\exp(-i\omega t)$ harmonic time-dependence is implicit throughout.

Without loss of generality, let $z \leq 0$ be the region of incidence and reflection, while $z \geq 0$ is the region of refraction. An arbitrarily-polarized incident plane-wave is specified by

$$E_{\text{inc}} = [A_1 u_y + A_2 (-\alpha_0 u_x + \kappa u_z)/k_0] \exp[i(\kappa x + \alpha_0 z)], \quad (1a)$$

$$\eta_0 H_{\text{inc}} = [-A_2 u_y + A_1 (-\alpha_0 u_x + \kappa u_z)/k_0] \cdot \exp[i(\kappa x + \alpha_0 z)], \quad (1b)$$

where

$$\kappa = k_0 \sin \theta_{\text{inc}}, \quad \alpha_0 = +\sqrt{(k_0^2 - \kappa^2)}, \quad (1c)$$

θ_{inc} is the angle of incidence with respect to the z axis, A_1 and A_2 are the known incidence coefficients, and u_x , etc., are the cartesian unit vectors.

Since Snel's laws are assumed *a priori* here, the reflected plane wave can be specified by

$$E_{\text{refl}} = [R_1 u_y + R_2 (\alpha_0 u_x + \kappa u_z)/k_0] \exp[i(\kappa x - \alpha_0 z)], \quad (2a)$$

$$\eta_0 H_{\text{refl}} = [-R_2 u_y + R_1 (\alpha_0 u_x + \kappa u_z)/k_0] \cdot \exp[i(\kappa x - \alpha_0 z)], \quad (2b)$$

while the refracted plane wave is given by

$$E_{\text{refr}} = [T_1 u_y + T_2 (-\alpha u_x + \kappa u_z)/k] \exp[i(\kappa x + \alpha z)], \quad (3a)$$

$$\eta H_{\text{refr}} = [-T_2 u_y + T_1 (-\alpha u_x + \kappa u_z)/k] \exp[i(\kappa x + \alpha z)], \quad (3b)$$

where

$$\alpha = +\sqrt{(k^2 - \kappa^2)}, \quad (3c)$$

R_1 and R_2 are the unknown reflection coefficients, and T_1 and T_2 are the unknown refraction coefficients.

By ensuring the continuity of the tangential components of the E and the H fields across the interface $z = 0$, the solution of this boundary value problem can be set down as [1]

$$R_1 = r_1 A_1, \quad T_1 = t_1 A_1, \quad R_2 = r_2 A_2, \quad T_2 = t_2 A_2, \quad (4)$$

in which r_1 and r_2 are the Fresnel reflection coefficients, while t_1 and t_2 are the Fresnel refraction (transmission) coefficients.

In most textbooks [e.g., 2, 3], the Fresnel coefficients are given in terms of the incidence angle θ_{inc} . Pertinent to the present context, however, it is useful to define a *bending* quantity

$$\xi_r = (\alpha/k)/(\alpha_0/k_0) \quad (5)$$

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which denotes the bending of the refracted plane wave with respect to the direction of propagation of the incident plane wave. The relative impedance

$$\eta_r = \eta / \eta_0 \quad (6)$$

is also defined.

In terms of ξ_r and η_r , the Fresnel coefficients can be specified as

$$r_1 = (\eta_r - \xi_r) / (\eta_r + \xi_r), \quad t_1 = 2\eta_r / (\eta_r + \xi_r), \quad (7)$$

and

$$r_2 = (1 - \eta_r \xi_r) / (1 + \eta_r \xi_r), \quad t_2 = 2\eta_r / (1 + \eta_r \xi_r). \quad (8)$$

From these expressions for the reflection coefficients, two critical ratios can be identified: $\eta_r \xi_r$ and η_r / ξ_r , besides the quantities η_r and ξ_r themselves. Bearing in mind that $|r_1| \leq 1$ and $|r_2| \leq 1$, the magnitudes of $\eta_r \xi_r$ and η_r / ξ_r may have three extreme values: 0, 1 and ∞ ; likewise, the magnitudes of η_r and ξ_r may also have the same three extreme values. These considerations lead to the following parametric conditions:

- (i) $\eta_r = \infty$, meaning that the medium of transmission is a perfect electric conductor;
- (ii) $\eta_r = 0$, meaning that the medium of transmission is a perfect magnetic conductor;
- (iii) $\xi_r = 0$, the case of total reflection because $\alpha = 0$;
- (iv) $\xi_r = \infty$, the case of grazing incidence because $\theta_{inc} = \pi/2$;
- (v) $\xi_r = 1/\eta_r$, meaning that $r_2 = 0$, which condition is satisfied at the Brewster angle of incidence provided $\epsilon_r > \mu_r$ [4,5]; and
- (vi) $\xi_r = \eta_r$, meaning that $r_1 = 0$, which condition is satisfied at the Brewster angle of incidence provided $\mu_r > \epsilon_r$ [4,5].

The simultaneous satisfaction of conditions (v) and (vi) implies that $\xi_r = \eta_r = 1$, which is possible only if both media are identical, and also suggests two pathological conditions, $\xi_r = 1$ and $\eta_r = 1$, that will now be explored. The reason for calling

these conditions *pathological* is because $|r_1| = |r_2|$ in both cases, while t_1 and t_2 are non-zero.

Pathological Condition 1: $\xi_r = 1$. In this case, $\alpha_o = \alpha$ so that $k_o = k$. Since the refractive indices of the two media are identical, it follows that $\epsilon_o \mu_o = \epsilon \mu$. Consequently, using $\xi_r = 1$ in (7) and (8) yields

$$r_1 = -r_2 = (\eta_r - 1)/(\eta_r + 1), \quad t_1 = t_2 = 2\eta_r/(\eta_r + 1). \quad (9)$$

Thus, the quantity $[(\epsilon_o \mu_o / \epsilon \mu) - 1]$ is not only a measure of the bending of the propagation direction of the refracted plane wave away from that of the incident plane wave, it is also a measure of the independence of the Fresnel coefficients from the angle of incidence, as has been pointed out by Giles and Wild [6].

Pathological Condition 2: $\eta_r = 1$. In this case, the two media have the same impedance; in other words, $\epsilon_o / \mu_o = \epsilon / \mu$. Consequently, from (7) and (8) it follows that

$$r_1 = r_2 = -(\xi_r - 1)/(\xi_r + 1), \quad t_1 = t_2 = 2/(\xi_r + 1), \quad (10)$$

as has also been observed by Giles and Wild [6].

It should be noted that real materials corresponding to either pathological condition may almost be impossible to find in nature. To explore the consequences of (10), one may turn to circularly polarized plane waves. Consider first the case of a right-circularly polarized (IEEE definition [7]) plane wave; then $A_2 = iA_1$. It follows from (4) and (10) that $R_2 = iR_1$ and $T_2 = iT_1$. Hence, the refracted, the reflected plane waves are also right circularly polarized. On the other hand, when $A_2 = -iA_1$, it can be seen from (4) and (10) that $R_2 = -iR_1$ and $T_2 = -iT_1$; hence, the incident, the reflected and the refracted plane waves are all left-circularly polarized. This aspect has been studied for magnetic spheres by Kerker *et al.* [8] and in a more general context by Lakhtakia *et al.* [9].

Before pressing on, it should be noted that the ratio of the reflected to the refracted power densities does not depend on the polarization state of the incident plane wave when either of the two pathological conditions hold. Further, the relation

$$(r_1 + r_2)/(1 + r_1 r_2) = (1 - \xi_r^2)/(1 + \xi_r^2) \quad (11)$$

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$$(r_1 - r_2)/(1 - r_1 r_2) = -(1 - \eta_r^2)/(1 + \eta_r^2) \quad (12)$$

that is independent of ξ_r , are to be noted. Both (11) and (12) describe hyperbolae in the r_1 - r_2 plane. Substitution of $\xi_r = 1$ in (11) or of $\eta_r = 1$ in (12) yields linear relationships between r_1 and r_2 . In other words, when either of the pathological conditions hold, the hyperbolae degenerate into straight lines.

It is instructive to compare the vibration ellipses of the three plane waves involved in the present problem. It is to be observed that

$$T_1 / T_2 = A_1 / A_2 \quad \text{for } \epsilon_o \mu_o = \epsilon \mu \text{ or } \epsilon_o / \mu_o = \epsilon / \mu, \quad (13a)$$

$$R_1 / R_2 = -A_1 / A_2 \quad \text{for } \epsilon_o \mu_o = \epsilon \mu, \quad (13b)$$

$$R_1 / R_2 = A_1 / A_2 \quad \text{for } \epsilon_o / \mu_o = \epsilon / \mu. \quad (13c)$$

As a result, the ellipticities of the three vibration ellipses are equal when either one of the pathological conditions holds.

Table 1: Vibration ellipse characteristics

Patho- logical Cond.	Plane Wave	Azi- muth†	Ellipti- city††	Handed- ness
	Inc.	ψ_{inc}	e_{inc}	h_{inc}
$\epsilon_o \mu_o = \epsilon \mu$	Refl.	ψ_{inc}	e_{inc}	$-h_{inc}$
$(\xi_r = 1)$	Refr.	ψ_{inc}	e_{inc}	h_{inc}
$\epsilon_o / \mu_o = \epsilon / \mu$	Refl.	$-\psi_{inc}$	e_{inc}	h_{inc}
$(\eta_r = 1)$	Refr.	ψ_{inc}	e_{inc}	h_{inc}

† The azimuth is sometimes called the *tilt*.

†† The ellipticity is often called the *axial ratio*.

The azimuth ψ [3, Sec. 1.4.2] of the refraction vibration ellipse is the same as that of the incidence ellipse for $\epsilon_0\mu_0 = \epsilon\mu$ or $\epsilon_0/\mu_0 = \epsilon/\mu$. When $\epsilon_0/\mu_0 = \epsilon/\mu$, the azimuth of the reflection ellipse is the negative of the incidence azimuth; but the two are the same for $\epsilon_0\mu_0 = \epsilon\mu$.

The handedness h of a plane wave with electric field E and propagation direction u_k is defined as [10]

$$h = i u_k \cdot (E \times E^*); \quad (14)$$

if $h > 0$ ($h < 0$), the wave is right-handed (left-handed). The handedness of the transmitted plane wave is the same as that of the incident plane wave provided $\epsilon_0\mu_0 = \epsilon\mu$ or $\epsilon_0/\mu_0 = \epsilon/\mu$. When $\epsilon_0\mu_0 = \epsilon\mu$, the handedness of the reflected wave is the negative of the incident plane wave; but the two handednesses are the same for $\epsilon_0/\mu_0 = \epsilon/\mu$. The foregoing conclusions have been tabulated in Table 1.

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