INPUT IMPEDANCE AND RESONANCES OF AN INFINITE L-C LADDER CIRCUIT

RICHARD S. ANDRULIS, JR.
Department of Engineering Science and Mechanics
The Pennsylvania State University
University Park, PA 16802
and
AKHLESH LAKHTAKIA
Center for the Engineering of Electronic and Acoustic Materials
Department of Engineering Science and Mechanics
The Pennsylvania State University
University Park, PA 16802

In recent years, continued fractions have been used for understanding such diverse topics as the fractal quantization of particles in one-dimensional potentials with incommensurate periods [1], the frustrated instabilities of active optical resonators [2], and the ac responses of rough surfaces [3, 4].

It is well-known that continued fractions can be used to represent ladder circuits [5], and that idea was utilized to explore the characteristics of resistance-capacitance and resistance-inductance circuits by Lakhtakia et al. [4]. This note examines the input impedance of an infinite ladder circuit, composed of purely reactive elements, in which scaling has been provided through a single parameter. It is shown that the input impedance does follow a scaling law. Further, the resonance frequencies of truncated circuits have been examined.

A continued fraction Q can be set up in terms of numbers $a_0, a_1, a_2, a_3,...$ as [6]

$$Q = \{a_0, a_1, a_2, a_3, ...\} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}} \tag{1}$$

One of the properties of an infinite continued fraction is that if it converges, any of its remainders (which are also infinite) also converge; that is if $\{a_0, a_1, a_2, a_3,...\}$ converges, then $\{a_1, a_2, a_3, a_4,...\}$ also converges.

Shown in Figure 1 is the ladder circuit studied here. In terms of the notation (1) for continued fractions, its input impedance $Z^{(\infty)}$ can be given as

$$Z^{(\infty)}(\omega) = \{j\omega L, j\omega C, j\omega L, a\, j\omega C, j\omega L, a^2\, j\omega C, j\omega L, a^3\, j\omega C, ...\}\ , \tag{2}$$

where L is the inductance, C is the capacitance, ω is the circular frequency, $j = \sqrt{(-1)}$, while a is the (real) scaling parameter. After some algebraic tedium, it can be easily shown that the following scaling relationships hold:

$$Z^{(\infty)}(a\,\omega) = \frac{Z^{(\infty)}(\omega) \cdot [\omega^2 LC - 1] + j\omega L \cdot [2 - \omega^2 LC]}{[j\omega C \cdot Z^{(\infty)}(\omega)] \cdot [1 + 1/a - \omega^2 LC] - 1/a + \omega^2 LC \cdot [1/a + 2 - \omega^2 LC]} \tag{3}$$

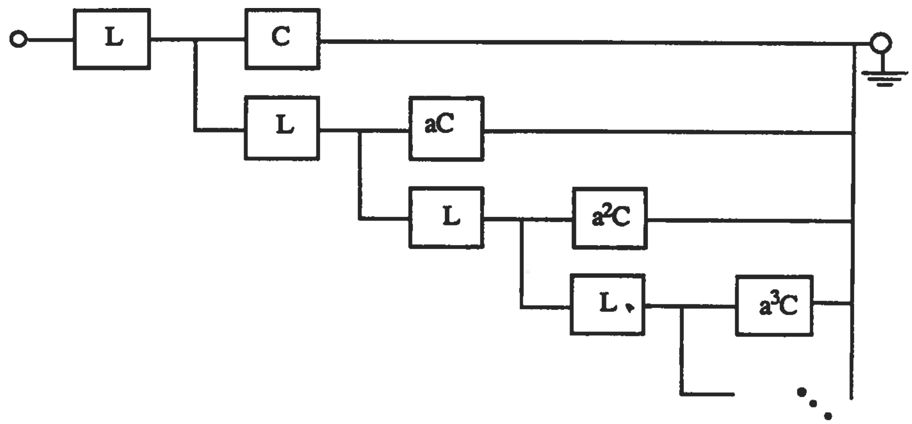

Figure 1 Schematic of the ladder L-C circuit.

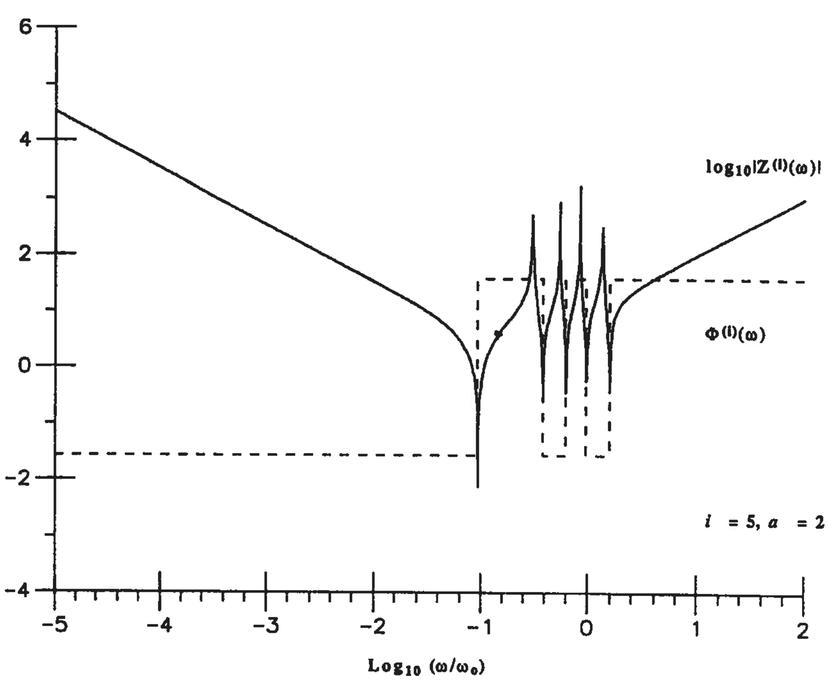

Figure 2 Plots of $\log_{10}|Z^{(i)}(\omega)|$ and $\varphi^{(i)}(\omega)$ versus $\log_{10}(\omega/\omega_o)$ for $i = 5$, $a = 2$, $L = 10$ and $C = 0.1$.

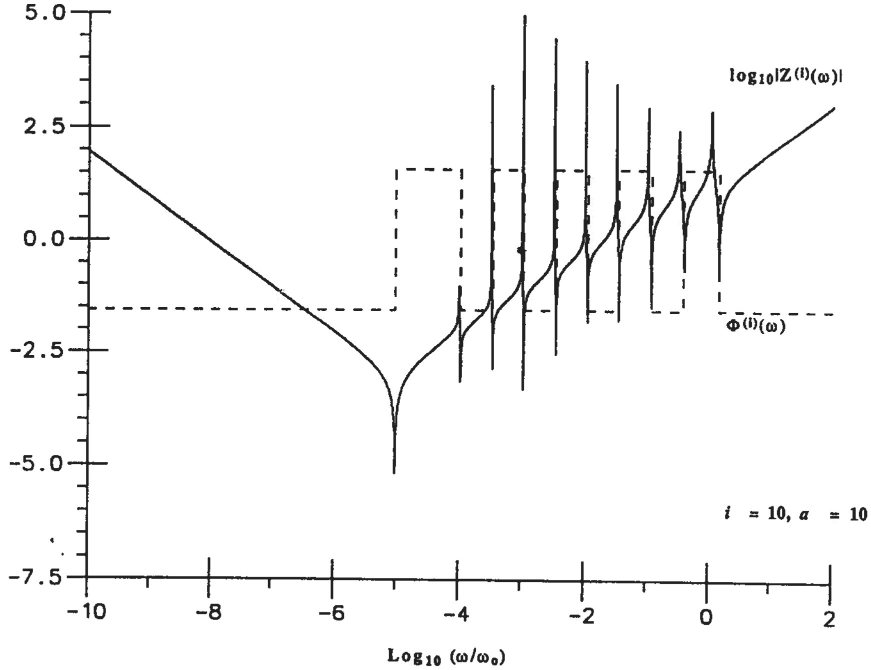

Figure 3 Plots of $\log_{10}|Z^{(i)}(\omega)|$ and $\varphi^{(i)}(\omega)$ versus $\log_{10}(\omega/\omega_o)$ for i = 10, a = 10, L = 10 and C = 0.1.

$$Z^{(\infty)}(\omega/a) = \frac{Z^{(\infty)}(\omega) \bullet [(\omega^2 LC)^2 - a\,\omega^2 LC - 2a^2\omega^2 LC + a^3] + j\omega L \bullet [2a^3 - a\,\omega^2 LC]}{a \bullet [Z^{(\infty)}(\omega) \bullet j\omega C \bullet [a^2 + a - \omega^2 LC] - a\,\omega^2 LC + a^3]} \quad (4)$$

It should be noted that the case $a = 1.0$ does not have any scaling.

Due to the presence of both inductors and capacitors in the circuit, resonances can occur. In order to investigate the resonance behavior, the circuit of Figure 1 was truncated, and the input impedance $Z^{(i)}(\omega)$ was defined as the finite-size continued fraction

$$Z^{(i)}(\omega) = \{j\omega L, j\omega C, j\omega L, a\,j\omega C, j\omega L, a^2\,j\omega C, \ldots\ldots, j\omega L, a^{i-1}\,j\omega C\} \quad . \quad (5)$$

Those frequencies at which $Z^{(i)}(\omega)$ is purely real constitute the resonance frequency of the truncated circuit.

At high frequencies, the capacitances are effectively short-circuits, while the inductances are open-circuits; consequently, $Z^{(i)}(\omega) \sim j\omega$ for large ω. The reverse is the case at low frequencies; hence $Z^{(i)}(\omega) \sim -j/\omega$ for small ω. Any resonance frequencies, therefore, can only be observed for intermediate values of ω.

Numerical computations of the resonance frequencies were made. As an example, in Figure 2 the magnitude $|Z^{(i)}(\omega)|$ and the phase $\varphi^{(i)}(\omega)$ are plotted for i = 5, a = 2, L = 10 and C = 0.1. Similarly, in Figure 3, the same quantities are plotted for i = 10, a = 10, L = 10 and C = 0.1. In these figures, the normalizing frequency $\omega_o = 1/\sqrt{(LC)}$. From the calculations reported in Figures 2 and 3, as well as from a host of other numerical studies, it was observed the $Z^{(i)}(\omega)$ has precisely i resonances. It was also observed that the largest resonance frequency does not exceed $2\omega_o$, although a lower bound for the resonance frequencies could not be identified.

Returning to $Z^{(i)}(\omega)$, it is noted that at small frequencies, the inductances can be effectively ignored, and the infinite-sized circuit is effectively capacitative; hence, in the limit $\omega \to 0$, the impedance will obey the approximate dynamic scaling law [7]

$$Z^{(\infty)}(\omega/a) \approx a\, Z^{(\infty)}(\omega) \quad . \tag{6}$$

This is, however, trivially so since all the inductance have been short-circuited and all the capacitances are in parallel. On the other hand, at large frequencies, the capacitances can be replaced by open-circuits, and the infinite-sized circuit is effectively inductive; hence, in the limit $\omega \rightarrow \infty$, the approximate scaling law

$$Z^{(\infty)}(\omega\, a) \approx a\, Z^{(\infty)}(\omega) \quad . \tag{7}$$

will be trivially obeyed since all the capacitances have been open-circuited and all of the inductances are in series.

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