A NOTE ON HUYGENS'S PRINCIPLE FOR UNIAXIAL DIELECTRIC MEDIA

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ABSTRACT

Three-dimensional dyadic Green's functions for homogeneous, uniaxial dielectric media are used to obtain mathematical statements of Huygens's principle pertaining to electric and magnetic fields in these media.

1. INTRODUCTION AND PRELIMINARIES

Seminal work on the electromagnetic (EM) theory of homogeneous, anisotropic media was carried out, among others, by Kong and Cheng during the 1960's, and a reasonably complete account is available in the books by Kong [1] and Chen [2]. Apart from the plasmas and ferrites, which have been studied and used for some time, a variety of anisotropic materials have found applications since then as substrates for microstrip antennas and integrated optical devices, for antenna radomes, and in the design of distributed active devices.

One of the most important facets of EM theory is the infinite-medium Green's function, which allows the solution of the Maxwell equations for diverse scattering and radiation problems. But anisotropic media lend themselves very well to problems involving cartesian geometries, and not too easily to cylindrical or spherical geometries. For this reason, time-harmonic Green's functions in the spherical co-ordinate system have been rarely investigated. the one notable exception are the recent derivations by Chen [3] and Lakhtakia et al., [4] for uniaxial dielectrics. In this paper, we derive mathematical statements of Huygens's principle for both the electric and the magnetic fields, using these Green's function.

Dielectric uniaxial media possess one optic axis, and are characterized by an anisotropic relative permittivity, given in dyadic notation as [2]

$$\overleftrightarrow{\epsilon} = \epsilon_{\perp} \overleftrightarrow{I} - [\epsilon_{\perp} - \epsilon_{||}]\, \mathbf{e}_z \mathbf{e}_z, \tag{1}$$

in which the optic axis runs parallel to the z axis without loss of generality, and $\overleftrightarrow{I}$ is the unit dyadic. Consequently, with the assumption of the harmonic time-dependence exp[-iωt], the latter two Maxwell's equations can be set down as

$$\nabla \times \mathbf{E} = i\omega\mu_o \mathbf{H} - \mathbf{K} \quad , \tag{2a}$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon_o \overleftrightarrow{\epsilon} \bullet \mathbf{E} + \mathbf{J}, \tag{2b}$$

in which ϵ_o and μ_o refer to free space, and **J** and **K** are the source terms. From these equations, the following Helmholtz-like wave equations can be derived:

$$\nabla \times \nabla \times \mathbf{E} - k_o{}^2 \overleftrightarrow{\epsilon} \bullet \mathbf{E} = i\omega\mu_o \mathbf{J} - \nabla \times \mathbf{K}, \tag{3a}$$

$$\nabla \times \overleftrightarrow{\epsilon}^{-1} \bullet \nabla \times \mathbf{H} - k_o{}^2 \mathbf{H} = \nabla \times \overleftrightarrow{\epsilon}^{-1} \bullet \mathbf{J} + i\omega\epsilon_o \mathbf{K} \quad . \tag{3b}$$

In these equations, $k_o{}^2 = \omega^2\epsilon_o\mu_o$ is the free space wavenumber, and

$$\overleftrightarrow{\epsilon}^{-1} = (1/\epsilon_{\perp}) \overleftrightarrow{I} - [\, (1/\epsilon_{\perp}) - (1/\epsilon_{||})\,]\, \mathbf{e}_z \mathbf{e}_z \tag{4}$$

is the dyadic inverse of $\overleftrightarrow{\epsilon}$. It is to be understood, here and hereafter, that the vector operations begin from the right-most one and proceed leftwards.

2. THE DYADIC GREEN'S FUNCTIONS

Because the medium is linear, the solutions of (3a,b) can be determined, respectively, as

$$\mathbf{E}(\mathbf{r}) = \iiint d^3r' \left[\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') \bullet [i\omega\mu_o \mathbf{J}(\mathbf{r}')] - \overleftrightarrow{G}_2(\mathbf{r},\mathbf{r}') \bullet \mathbf{K}(\mathbf{r}') \right] \quad , \tag{5a}$$

$$\mathbf{H}(\mathbf{r}) = \iiint d^3r' \left[\overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') \bullet \mathbf{J}(\mathbf{r}') + i\omega\epsilon_o \overleftrightarrow{G}_4(\mathbf{r},\mathbf{r}') \bullet \mathbf{K}(\mathbf{r}') \right], \tag{5b}$$

in which the four dyadic Green's functions satisfy the following equations:

$$[\nabla \times \nabla \times \overleftrightarrow{I} - k_o{}^2 \overleftrightarrow{\epsilon}] \bullet \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') = \overleftrightarrow{I}\delta(\mathbf{r}-\mathbf{r}'), \tag{6a}$$

$$[\nabla \times \nabla \times \overleftrightarrow{I} - k_o{}^2 \overleftrightarrow{\epsilon}] \bullet \overleftrightarrow{G}_2(\mathbf{r},\mathbf{r}') = \nabla \times \overleftrightarrow{I}\delta(\mathbf{r}-\mathbf{r}') \quad , \tag{6b}$$

$$[\nabla \times \overleftrightarrow{\epsilon}^{-1} \bullet \nabla \times \overleftrightarrow{I} - k_o{}^2 \overleftrightarrow{I}] \bullet \overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') = \nabla \times \overleftrightarrow{\epsilon}^{-1}\, \delta(\mathbf{r}-\mathbf{r}'), \tag{6c}$$

$$[\nabla \times \overleftrightarrow{\epsilon}^{-1} \bullet \nabla \times \overleftrightarrow{I} - k_o{}^2 \overleftrightarrow{I}] \bullet \overleftrightarrow{G}_4(\mathbf{r},\mathbf{r}') = \overleftrightarrow{I}\delta(\mathbf{r}-\mathbf{r}') \quad . \tag{6d}$$

Rather simple manipulations of these equations yield the relationships

$$\overleftrightarrow{G}_2(\mathbf{r},\mathbf{r}') = \overleftrightarrow{\epsilon}^{-1} \bullet \nabla \times \overleftrightarrow{G}_4(\mathbf{r},\mathbf{r}'), \qquad \overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') = \nabla \times \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}'); \tag{7a,b}$$

hence, only the solutions of (6a) and (6d) are strictly necessary.

Chen [2] has derived an expression for the electric Green's dyadic $\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}')$, which is re-stated here for convenience:

$$\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') = [\epsilon_\perp \epsilon_{||} \overleftrightarrow{\epsilon}^{-1} + \nabla\nabla/k_o{}^2]\, g_e/4\pi\sqrt{(\epsilon_\perp)} + \overleftrightarrow{X}(\mathbf{r}-\mathbf{r}') \quad , \tag{8}$$

where the dyadic $\overleftrightarrow{X}(\mathbf{r},\mathbf{r}')$ is given as

$$4\pi\sqrt{(\epsilon_\perp)}\, \overleftrightarrow{X}(\mathbf{R}) = (\epsilon_\perp g_o - \epsilon_{||} g_e) \left[(\mathbf{R}\times\mathbf{e}_z)(\mathbf{R}\times\mathbf{e}_z)/(\mathbf{R}\times\mathbf{e}_z)^2 \right]$$
$$+ (R_o g_o - R_e g_e) \left[\overleftrightarrow{I} - \mathbf{e}_z\mathbf{e}_z - 2(\mathbf{R}\times\mathbf{e}_z)(\mathbf{R}\times\mathbf{e}_z)/(\mathbf{R}\times\mathbf{e}_z)^2 \right] / ik_o(\mathbf{R}\times\mathbf{e}_z)^2 \quad . \tag{9}$$

Using (7b) and (8), the dyadic $\overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}')$, can be derived out to be

$$4\pi\sqrt{(\epsilon_\perp)}\, \overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') = \epsilon_\perp (g_e - g_o)(\mathbf{R} \bullet \mathbf{e}_z)[\mathbf{e}_z\times(\mathbf{R}\times\mathbf{e}_z)][\mathbf{R}\times\mathbf{e}_z] / (\mathbf{R}\times\mathbf{e}_z)^4 +$$
$$+ \epsilon_\perp (g_e - g_o)(\mathbf{R} \bullet \mathbf{e}_z)\, [\mathbf{R}\times\mathbf{e}_z]\, [\mathbf{e}_z\times(\mathbf{R}\times\mathbf{e}_z)] / (\mathbf{R}\times\mathbf{e}_z)^4 -$$
$$- \epsilon_\perp{}^2 g_o R_o{}^{-2} (1 - ik_o R_o)\, [\mathbf{R}\times(\mathbf{R}\times\mathbf{e}_z)][\mathbf{R}\times\mathbf{e}_z] / (\mathbf{R}\times\mathbf{e}_z)^2 +$$
$$+ \epsilon_\perp \epsilon_{||}\, g_e R_e{}^{-2} (1 - ik_o R_e)\, [\mathbf{R}\times\mathbf{e}_z][\mathbf{R}\times(\mathbf{R}\times\mathbf{e}_z)] /$$
$$(\mathbf{R}\times\mathbf{e}_z)^2 \quad . \tag{10}$$

In these equations, as well as hereafter, $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ and the following notation holds:

$$g_o = \exp[ik_o R_o]/R_o \quad , \tag{11a}$$

$$g_e = \exp[ik_o R_e]/R_e \ , \tag{11b}$$

$$R_o = \sqrt{[\varepsilon_\perp \mathbf{R} \bullet \mathbf{R}]} \quad , \tag{11c}$$

$$R_e = \sqrt{[\varepsilon_{||}(\mathbf{R}\times\mathbf{e}_z)^2 + \varepsilon_\perp (\mathbf{R} \bullet \mathbf{e}_z)^2]} . \tag{11d}$$

The derivation of the magnetic Green's dyadic $\overleftrightarrow{G}_4 =(\mathbf{r}, \mathbf{r}')$ is due to Lakhtakia et al. [4], who obtained it as

$$\overleftrightarrow{G}_4 (\mathbf{r},\mathbf{r}') = \varepsilon_\perp \{[\varepsilon_\perp \overleftrightarrow{I} + \nabla\nabla/k_o^{\ 2}]\, g_o/4\mu\pi\sqrt{(\varepsilon_\perp)} - \overleftrightarrow{X}(\mathbf{r} - \mathbf{r}')\} \ . \tag{12}$$

Finally, using (7a) and (12), the remaining dyadic $\overleftrightarrow{G}_2(\mathbf{r},\mathbf{r}')$ can be otained as

$$\begin{aligned} 4\pi\sqrt{(\varepsilon_\perp)}\, \overleftrightarrow{G}_2 (\mathbf{r},\mathbf{r}') = {} & \varepsilon_\perp (g_e - g_o)(\mathbf{R} \bullet \mathbf{e}_z)[\mathbf{e}_z\times(\mathbf{R}\times\mathbf{e}_z)][\mathbf{R}\times\mathbf{e}_z] / (\mathbf{R}\times\mathbf{e}_z)^4 + \\ & + \varepsilon_\perp (g_e - g_o)(\mathbf{R} \bullet \mathbf{e}_z)\, [\mathbf{R}\times\mathbf{e}_z]\, [\mathbf{e}_z\times(\mathbf{R}\times\mathbf{e}_z)] / (\mathbf{R}\times\mathbf{e}_z)^4 - \\ & - \varepsilon_\perp^{\ 2}\, g_o\, R_o^{\ -2} (1 - ik_o R_o)\, [\mathbf{R}\times\mathbf{e}_z]\, [\mathbf{R}\times(\mathbf{R}\times\mathbf{e}_z)] / (\mathbf{R}\times\mathbf{e}_z)^2 + \\ & + \varepsilon_\perp \varepsilon_{||}\, g_e\, R_e^{\ -2} (1 - ik_o R_e)\, [\mathbf{R}\times(\mathbf{R}\times\mathbf{e}_z)]\, [\mathbf{R}\times\mathbf{e}_z] / (\mathbf{R}\times\mathbf{e}_z)^2 \quad , \end{aligned} \tag{13}$$

The uniaxial medium is reciprocal; this fact is reflected by the transpose properties of the Green's functions. Inspection of (8), (10), (12) and (13) quickly reveals that

$$[\overleftrightarrow{G}_1 (\mathbf{r},\mathbf{r}')]^T = \overleftrightarrow{G}_1 (\mathbf{r}',\mathbf{r}) \quad , \tag{14a}$$

$$[\overleftrightarrow{G}_2 (\mathbf{r},\mathbf{r}')]^T = \overleftrightarrow{G}_3 (\mathbf{r}',\mathbf{r}) \quad , \tag{14b}$$

$$[\overleftrightarrow{G}_3 (\mathbf{r},\mathbf{r}')]^T = \overleftrightarrow{G}_2 (\mathbf{r}',\mathbf{r}) , \tag{14c}$$

$$[\overleftrightarrow{G}_4 (\mathbf{r},\mathbf{r}')]^T = \overleftrightarrow{G}_4 (\mathbf{r}',\mathbf{r}) \quad , \tag{14d}$$

the superscript 'T' denoting the transpose. Furthermore, the following symmetry properties also hold:

$$\overleftrightarrow{G}_1 (\mathbf{r},\mathbf{r}') = \overleftrightarrow{G}_1 (\mathbf{r}',\mathbf{r}) \ , \tag{15a}$$

$$\overleftrightarrow{G}_2 (\mathbf{r},\mathbf{r}') = - \overleftrightarrow{G}_2 (\mathbf{r}',\mathbf{r}) , \tag{15b}$$

$$\overleftrightarrow{G}_3 (\mathbf{r},\mathbf{r}') = - \overleftrightarrow{G}_3 (\mathbf{r}',\mathbf{r}) , \tag{15c}$$

$$\overleftrightarrow{G}_4 (\mathbf{r},\mathbf{r}') = \overleftrightarrow{G}_4 (\mathbf{r}',\mathbf{r}) \ . \tag{15d}$$

3. HUYGENS'S PRINCIPLE

With the help of the Green's dyadics, we are ready to derive mathematical statements of Huygens's principle pertaining to radiation and scattering in uniaxial media. Consider a (uniaxial dielectric) volume V_e which lies in between the S and S_∞, as shown in Fig. 1. Then the vector-dyadic Green's theorem [5] can be applied to yield

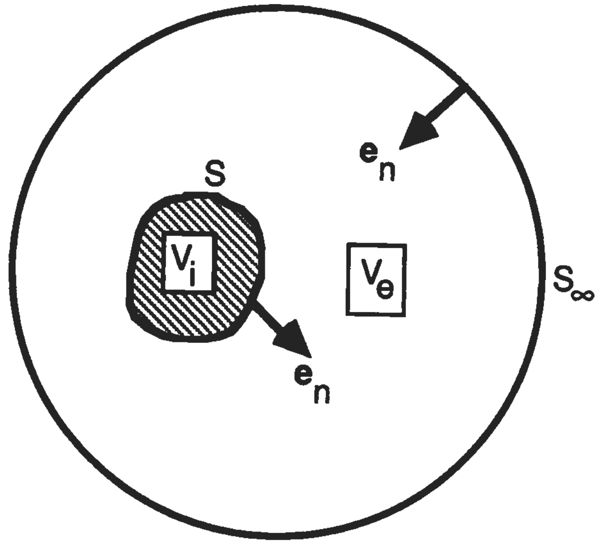

Figure 1 Relevant to the development of Huygens's Principle.

$$\iiint_{V_e} d^3r \left\{ \mathbf{E}(\mathbf{r}) \bullet [\nabla\times\nabla\times\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}')] - [\nabla\times\nabla\times\mathbf{E}(\mathbf{r})] \bullet \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') \right\} =$$

$$\iint_{S+S_\infty} d^2r \left\{ [\mathbf{e}_n\times(\nabla\times\mathbf{E}(\mathbf{r})] \bullet \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') - [\mathbf{e}_n\times\mathbf{E}(\mathbf{r})] \bullet [\nabla\times\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}')] \right\} . \quad (16)$$

There are no sources in V_e, and (3a) as well as (6a) can be utilized, along with the symmetry of the constitutive dyadic $\overleftrightarrow{\epsilon}$, to yield

$$\mathbf{E}(\mathbf{r}') = \iint_{S+S_\infty} d^2r \left\{ [\mathbf{e}_n\times(\nabla\times\mathbf{E}(\mathbf{r})] \bullet \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') - [\mathbf{e}_n\times\mathbf{E}(\mathbf{r})] \bullet [\nabla\times\overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') \right\} ;$$

$$\mathbf{r}' \in V_e \quad . \quad (17)$$

Next, the integral on the surface S_∞ can be eliminated by virtue of the satisfaction of appropriate radiation conditions by the Green's functions. Then, (7b) as well as (2a) can be utilized in (17) to give

$$\mathbf{E}(\mathbf{r}') = \iint_S d^2r \left\{ i\omega\mu_o [\mathbf{e}_n\times\mathbf{H}(\mathbf{r})] \bullet \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') - [\mathbf{e}_n\times\mathbf{E}(\mathbf{r})] \bullet \overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') \right\} ; \mathbf{r}' \in V_e \quad . \quad (18)$$

On interchanging the primed and the unprimed variables, and utilizing the transpose properties of $\overleftrightarrow{G}_1$ and $\overleftrightarrow{G}_3$, *Huygens's principle for the electric field* can be stated as

$$\mathbf{E}(\mathbf{r}) = \iint_S d^2r' \left\{ i\omega\mu_o \overleftrightarrow{G}_1(\mathbf{r},\mathbf{r}') \bullet [\mathbf{e}_n\times\mathbf{H}(\mathbf{r}')] + \overleftrightarrow{G}_2(\mathbf{r},\mathbf{r}') \bullet [\mathbf{e}_n\times\mathbf{E}(\mathbf{r}')] \right\} ; \mathbf{r} \in V_e \quad . \quad (19)$$

Next, by taking the curl ($\nabla\times$) of both sides of (19), and after using (7a,b), one obtains

$$\nabla\times\mathbf{E}(\mathbf{r}) = \iint_S d^2r' \left\{ i\omega\mu_o \overleftrightarrow{G}_3(\mathbf{r},\mathbf{r}') \bullet [\mathbf{e}_n\times\mathbf{H}(\mathbf{r}')] + \right.$$

$$\left. + [\nabla\times\overleftrightarrow{\epsilon}^{-1} \bullet \nabla\times\overleftrightarrow{G}_4(\mathbf{r},\mathbf{r}')] \bullet [\mathbf{e}_n\times\mathbf{E}(\mathbf{r}')] \right\} ; \ \mathbf{r} \in V_e . \quad (20)$$

It should be noted that V_e is source-free, therefore by additionally restricting $\mathbf{r} \notin S$, and using (2a) and (6d), *Huygens's principle for the magnetic field* is obtained as

$$\mathbf{H}(\mathbf{r}) = \iint_S d^2\mathbf{r}' \Big\{ \overleftrightarrow{\mathbf{G}}_3(\mathbf{r},\mathbf{r}') \bullet [\mathbf{e}_n \times \mathbf{H}(\mathbf{r}')] - \\ -i\omega\epsilon_o \overleftrightarrow{\mathbf{G}}_4(\mathbf{r},\mathbf{r}') \bullet [\mathbf{e}_n \times \mathbf{E}(\mathbf{r}')] \Big\} ; \mathbf{r} \notin S, \mathbf{r} \in V_e \quad . \tag{21}$$

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