Cartesian Solutions of Maxwell's Equations for Linear, Anisotropic Media — Extension of the Mrozowski Algorithm

The Mrozowski algorithm for finding cartesian solutions of Maxwell's equations for anisotropic media ($\vec{D} = \varepsilon_0 \overset{\leftrightarrow}{\varepsilon} \vec{E}$, $\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H}$) is extended to even a more general class of linear, homogeneous, anisotropic media ($\vec{D} = \varepsilon_0 \overset{\leftrightarrow}{\varepsilon} \vec{E} - j\eta_0^{-1} \cdot \mu_0 \overset{\leftrightarrow}{r} \vec{H}$, $\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H} + j\eta_0 \varepsilon_0 \overset{\leftrightarrow}{s} \vec{E}$). With the presented extension, the utility of the Mrozowski algorithm has been greatly enhanced.

Cartesische Lösungen der Maxwellschen Gleichungen für lineare anisotrope Medien — Erweiterung des Mrozowski Algorithmus

Der Algorithmus von Mrozowski zur Bestimmung cartesischer Lösungen der Maxwellschen Gleichungen für anisotrope Medien ($\vec{D} = \varepsilon_0 \overset{\leftrightarrow}{\varepsilon} \vec{E}$, $\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H}$) wird auf eine allgemeinere Klasse von linearen, homogenen, anisotropen Medien ($\vec{D} = \varepsilon_0 \overset{\leftrightarrow}{\varepsilon} \vec{E} - j\eta_0^{-1} \mu_0 \overset{\leftrightarrow}{r} \vec{H}$, $\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H} + j\eta_0 \varepsilon_0 \overset{\leftrightarrow}{s} \vec{E}$) erweitert. Durch diese Erweiterung wird die Anwendbarkeit des Mrozowski Algorithmus wesentlich verbreitert.

1. Introduction

In a recent communication to AEÜ, Mrozowski [1] has described a new computer-oriented approach for finding out cartesian solutions of Maxwell's equations in a bianisotropic medium in which the constitutive equations

$$\vec{D} = \varepsilon_0 \overset{\leftrightarrow}{\varepsilon} \vec{E} \tag{1a}$$

$$\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H} \tag{1b}$$

hold, with $\overset{\leftrightarrow}{\varepsilon}$ and $\overset{\leftrightarrow}{\mu}$ being, in general, full, complex tensors of rank 3. Adhering as far as possible to

Mrozowski's notation, it is assumed in [1] that the electromagnetic field can be decomposed into the form

$$\vec{E}(x,y,z) = (2\pi)^{-2} k_0 \int\limits_{-\infty}^{\infty} d\beta_n \tag{2a}$$

$$\cdot \exp[-jk_0\beta_n z] \int\limits_{-\infty}^{\infty} dp \, \exp[jpy] \, \vec{e}(x,p,\beta_n),$$

$$\vec{H}(x,y,z) = (2\pi)^{-2} k_0 \int\limits_{-\infty}^{\infty} d\beta_n \cdot \tag{2b}$$

$$\cdot \exp[-jk_0\beta_n z] \int\limits_{-\infty}^{\infty} dp \, \exp[jpy] \, \vec{h}(x,p,\beta_n).$$

These decompositions are then substituted into Maxwell's equations along with the constitutive equations (1), and eventually, after eliminating e_x and h_x, a matrix equation is arrived at relating the field components e_y, e_z, h_y and h_z to their x-derivatives. The solution of this matrix equation, eq. (3) of [1], can be attempted on a digital computer; as a result solutions for Maxwell's equations applicable to the full tensor medium described by (1) can be obtained using widely available software library routines.

Here, I wish to point out that the Mrozowski algorithm is even more widely applicable. Assuming a general linear field structure [2], where the fields $\vec{D}$ and $\vec{B}$ are dependent on both $\vec{E}$ and $\vec{H}$, this algorithm enables the analysis of a wide variety of generally anisotropic media, including, but not con-

fined to, gyroelectric [3] and gyromagnetic [4] media, isotropic chiral media [5], and various classes of crystals [6]–[8]. Such media are being increasingly explored because of potential use in hybrid and monolithic integrated circuits as well as in sensor technology, and a good recent and relevant review is contained in a paper by Krowne [9].

2. Analysis and Discussion

The constitutive equations for a general, linear, homogeneous, anisotropic medium can be formally expressed as [1], [9]

$$\vec{D} = \varepsilon_0 \, \overset{\leftrightarrow}{\varepsilon} \, \vec{E} - \mathrm{j} \, \eta_0^{-1} \mu_0 \overset{\leftrightarrow}{r} \, \vec{H}, \tag{3a}$$

$$\vec{B} = \mu_0 \overset{\leftrightarrow}{\mu} \vec{H} + \mathrm{j} \, \eta_0 \, \varepsilon_0 \overset{\leftrightarrow}{s} \, \vec{E}. \tag{3b}$$

Substitution of (3) in the Maxwell's equations for curl $\vec{E}$ and curl $\vec{H}$ leads to

$$\mathrm{curl}\, \vec{E} = k_0 [\overset{\leftrightarrow}{\mu} \, \eta \, \vec{H} + \overset{\leftrightarrow}{s} \, \vec{E}], \tag{4a}$$

$$\mathrm{curl}\, \eta \vec{H} = k_0 [\overset{\leftrightarrow}{r} \, \eta \, \vec{H} + \overset{\leftrightarrow}{\varepsilon} \, \vec{E}], \tag{4b}$$

where $\eta = -\mathrm{j}\,\eta_0$, and η_0 and k_0 are the intrinsic impedance of and the wavenumber in free space, respectively. By substituting the two decompositions (2) in (4a, b), it is easy to show that

$$-k_0 \boldsymbol{P} \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xt} & r_{xx} & r_{xt} - k \\ \varepsilon_{tx}^{\mathrm{T}} & \varepsilon_{tt} & (r_{tx}+k)^{\mathrm{T}} & r_{tt} \\ s_{xx} & s_{xt} - k & \mu_{xx} & \mu_{xt} \\ (s_{tx}+k)^{\mathrm{T}} & s_{tt} & \mu_{tx}^{\mathrm{T}} & \mu_{tt} \end{pmatrix} \begin{pmatrix} e_x \\ e_t^{\mathrm{T}} \\ \eta\, h_x \\ \eta\, \boldsymbol{h}_t^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} 0 \\ \eta \{\mathrm{d}/\mathrm{d}x\} \, (h_z, h_y)^{\mathrm{T}} \\ 0 \\ \{\mathrm{d}/\mathrm{d}x\} \, (e_z, e_y)^{\mathrm{T}} \end{pmatrix} \tag{5}$$

where the 6×6 matrix $\boldsymbol{P} = \mathrm{diag}\,(1, 1, -1, 1, 1, -1)$; the 1×2 row vectors $\boldsymbol{k} = (\mathrm{j}\,\beta_n, \mathrm{j}\, p/k_0)$, $\boldsymbol{e}_t = (e_y, e_z)$ and $\boldsymbol{h}_t = (h_y, h_z)$, with the superscript 'T' denoting the transpose. The constitutive tensors are partitioned as shown below,

$$\overset{\leftrightarrow}{\varepsilon} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xt} \\ \varepsilon_{tx}^{\mathrm{T}} & \varepsilon_{tt} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix} \tag{6}$$

with ε_{xx} being a scalar, ε_{tx} and ε_{xt} being 1×2 row vectors, and ε_{tt} is a 2×2 matrix.

Eq. (5) can be further simplified by eliminating e_x and ηh_x from it. In the manner of Mrozowski [1], one obtains

$$\{\mathrm{d}/\mathrm{d}x\} \, (\boldsymbol{e}_t, \eta\, \boldsymbol{h}_t)^{\mathrm{T}} = \boldsymbol{R}\, \boldsymbol{Q}\, (\boldsymbol{e}_t, \eta\, \boldsymbol{h}_t)^{\mathrm{T}}, \tag{7}$$

which is *exactly* like eq. (3) of [1], and which is solved like a matrix eigenvalue problem by his algorithm. In (7), $\boldsymbol{R}$ is a 4×4 matrix whose right-to-left diagonal elements are all unity, the remaining elements being identically zero. The 2×2 submatrix elements of the 4×4 matrix $\boldsymbol{Q}$ are given by

$$\boldsymbol{Q}_{11} = \boldsymbol{K}_0 \, g_{xx} \{[\mu_{xx}\, \varepsilon_{tx}^{\mathrm{T}} - s_{xx}\, (r_{tx}+k)^{\mathrm{T}}]\, \varepsilon_{xt}$$
$$- [r_{xx}\, \varepsilon_{tx}^{\mathrm{T}} - \varepsilon_{xx}\, (r_{tx}+k)^{\mathrm{T}}]\, (s_{xt} - k)\} + \boldsymbol{K}_0\, \varepsilon_{tt},$$

$$\boldsymbol{Q}_{12} = \boldsymbol{K}_0 \, g_{xx} \{[\mu_{xx}\, \varepsilon_{tx}^{\mathrm{T}} - s_{xx}\, (r_{tx}+k)^{\mathrm{T}}]\, (r_{xt} - k) -$$
$$- [r_{xx}\, \varepsilon_{tx}^{\mathrm{T}} - \varepsilon_{xx}\, (r_{tx}+k)^{\mathrm{T}}]\, \mu_{xt}\} + \boldsymbol{K}_0\, r_{tt},$$

$$\boldsymbol{Q}_{21} = \boldsymbol{K}_0 \, g_{xx} \{[\mu_{xx}\, (s_{tx}+k)^{\mathrm{T}} - s_{xx}\, \mu_{tx}^{\mathrm{T}}]\, \varepsilon_{xt} -$$
$$- [r_{xx}\, (s_{tx}+k)^{\mathrm{T}} - \varepsilon_{xx}\, \mu_{tx}^{\mathrm{T}}]\, (s_{xt} - k)\} + \boldsymbol{K}_0\, s_{tt},$$

$$\boldsymbol{Q}_{22} = \boldsymbol{K}_0 \, g_{xx} \{[\mu_{xx}\, (s_{tx}+k)^{\mathrm{T}} - s_{xx}\, \mu_{tx}^{\mathrm{T}}]\, (r_{xt} - k) -$$
$$- [r_{xx}\, (s_{tx}+k)^{\mathrm{T}} - \varepsilon_{xx}\, \mu_{tx}^{\mathrm{T}}]\, \mu_{xt}\} + \boldsymbol{K}_0\, \mu_{tt}, \tag{8}$$

with $g_{xx}(s_{xx}\, r_{xx} - \varepsilon_{xx}\, \mu_{xx}) = 1$, and the 2×2 matrix $\boldsymbol{K}_0 = \mathrm{diag}\,(-k_0, k_0)$. It should be noted that by setting the tensors $\overset{\leftrightarrow}{r}$ and $\overset{\leftrightarrow}{s}$ to zero, the definitions (8) reduce to the definitions (4) of [1].

As a result of the foregoing discussion, the Mrozowski algorithm for finding the cartesian solutions of Maxwell's equation subject to the constitutive equations (1) is also perfectly capable of handling the even more general case [2] when the constitutive equations (3) apply. As such, the procedure is valid for exploring the dielectric Faraday effect [3], the magnetic Faraday effect [4], natural optical activity [5] and crystals [6]–[8], among several other possibilities. This generality of the Mrozowski algorithm makes it very useful indeed.

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